XXVIII. *Dynamical Problems regarding Elastic Spheroidal Shells and Spheroids of Incompressible Liquid.* By *Professor* W. Thomson, *LL.D., F.R.S.*

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1. The theory of elastic solids in equilibrium presents the following general problem:—

A solid of any shape being given, and displacements being arbitrarily produced or forces arbitrarily applied over its whole bounding surface, it is required to find the displacement of every point of its substance. The chief object of the present communication is to show the solution of this problem for the case of a shell consisting of isotropic elastic material, and bounded by two concentric spherical surfaces, with the natural restriction that the whole alteration of figure is very small.

2. Let the centre of the spherical surfaces be taken as origin, and let x, y, z be the rectangular coordinates of any particle of the solid, in its undisturbed position, and $x+\alpha$, $y+\beta$, $z+\gamma$ the coordinates of the same particle when the whole is in equilibrium under the given superficial disturbing action. Then, by the known equations of equilibrium of elastic solids, we have

$$\left.\begin{array}{l} n\left(\frac{d^2\alpha}{dx^2}+\frac{d^2\alpha}{dy^2}+\frac{d^2\alpha}{dz^2}\right)+m\frac{d}{dx}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=0, \\ n\left(\frac{d^2\beta}{dx^2}+\frac{d^2\beta}{dy^2}+\frac{d^2\beta}{dz^2}\right)+m\frac{d}{dy}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=0, \\ n\left(\frac{d^2\gamma}{dx^2}+\frac{d^2\gamma}{dy^2}+\frac{d^2\gamma}{dz^2}\right)+m\frac{d}{dz}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=0, \end{array}\right\} \quad \ldots\ldots\ldots \quad (1)$$

$m-\frac{1}{3}n$ and n denoting the two coefficients of elasticity, which may be called respectively the *elasticity of volume*, and the *rigidity*. A demonstration of these equations, with definitions of the coefficients, will be found in § 71 of an Appendix to the present communication.

3. For brevity let

$$\delta=\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}, \quad \ldots\ldots\ldots \quad (2)$$

so that δ shall denote the cubic dilatation at the point (x, y, z) of the solid. Also, for brevity, let the operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$ be denoted by ∇^2. Then the preceding equations become

$$\left.\begin{array}{l} n\nabla^2\alpha+m\frac{d\delta}{dx}=0, \\ n\nabla^2\beta+m\frac{d\delta}{dy}=0, \\ n\nabla^2\gamma+m\frac{d\delta}{dz}=0. \end{array}\right\} \quad \ldots\ldots\ldots \quad (3)$$

4. In certain cases, especially the ideal one of an incompressible elastic solid, the following notation is more convenient:—

p the mean normal pressure per unit of area on all sides of any small portion of the solid, round the point x, y, z. Then (below, § 21)

$$p=-(m-\tfrac{1}{3}n)\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right); \quad \ldots \ldots \quad (4)$$

and the equations of equilibrium become

$$\left.\begin{aligned} n\nabla^2\alpha-\frac{m}{m-\frac{1}{3}n}\frac{dp}{dx}&=0,\\ n\nabla^2\beta-\frac{m}{m-\frac{1}{3}n}\frac{dp}{dy}&=0,\\ n\nabla^2\gamma-\frac{m}{m-\frac{1}{3}n}\frac{dp}{dz}&=0.\end{aligned}\right\} \quad \ldots \ldots \quad (5)$$

5. If the solid were incompressible, we should have $m=\infty$ and

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0,$$

which must be taken instead of (4), and, along with (5), would constitute the four differential equations required for the four unknown functions α, β, γ, p *.

6. To solve the general equations (3) or (5), take $\frac{d}{dx}$ of the first, $\frac{d}{dy}$ of the second, and $\frac{d}{dz}$ of the third; and add. We have thus

$$(n+m)\nabla^2\delta=0, \quad \ldots \ldots \quad (6)$$

or, which is in general sufficient,

$$\nabla^2\delta=0. \quad \ldots \ldots \quad (7)$$

If, now, an appropriate solution of this equation for δ is found, the three equations (3) may be solved by known methods, the first of them for α, the second for β, and the third for γ,—the arbitrary part of the solution in each case being merely a solution of the equation $\nabla^2u=0$. These arbitrary parts must be determined so as to fulfil equation (2) and the prescribed surface conditions.

The complete particular determination of δ cannot, however, in most cases be effected without regard to α, β, γ; and the order of procedure which has been indicated is only convenient for determining the proper forms for general solutions of the equations.

7. First, then, to solve the equation in δ generally, we may use a theorem belonging to the foundation of LAPLACE's remarkable analysis of the attraction of spheroids, which may be enunciated as follows.

If the equation $\nabla^2\delta=0$ is satisfied for every point between two concentric spheres of

* See Professor STOKES's paper "On the Friction of Fluids in Motion, and the Equilibrium and Motion of Elastic Solids," Cambridge Philosophical Society's Transactions, April, 1845.

radii a (greater) and a' (less), the value of δ for any point of this space, at distance r from the centre, may be expressed by the double series

$$\begin{array}{l} V_0 \quad +V_1 \quad +V_2 \quad + \&c. \\ +V'_0 r^{-1}+V'_1 r^{-3}+V'_2 r^{-5}+ \&c., \end{array}$$

of which the first part converges at least as rapidly as the geometrical progression

$$\frac{r}{a}, \left(\frac{r}{a}\right)^2, \left(\frac{r}{a}\right)^3, \ldots$$

and the second at least as rapidly as

$$\frac{a'}{r}, \left(\frac{a'}{r}\right)^2, \left(\frac{a'}{r}\right)^3, \ldots,$$

—if V_i, V'_i denote homogeneous functions of x, y, z of the order i, each satisfying, continuously, for all values of x, y, z, the equation

$$\nabla^2 V=0.$$

A proof of this proposition is given in THOMSON and TAIT'S 'Natural Philosophy,' chap. i. Appendix B. It is also there shown, what I believe has been hitherto overlooked, that V_i, V'_i, as above defined, cannot but be rational and integral, if i is any positive integer.

8. To avoid circumlocution, we shall call any homogeneous function of (x, y, z) which satisfies the equation

$$\nabla^2 V=0$$

a "spherical harmonic function," or, more shortly, a "spherical harmonic." Thus V_i and V'_i, as defined in § 7, are spherical harmonics of degree or order i; and $V'_i r^{-2i-1}$, being also a solution of $\nabla^2 V=0$, is a spherical harmonic of degree $-(i+1)$. We shall sometimes call the latter a spherical harmonic of inverse order i. Thus u_i being any spherical harmonic of integral degree i, and therefore necessarily a rational integral function of this degree, $u_i r^{-2i-1}$ is a spherical harmonic of degree $-(i+1)$, or of inverse order i.

If we put $-(i+1)=j$, and denote this last function by ϕ_j, then we have

$$\phi_j r^{-2j-1}=u_i;$$

and thus it appears that the relation between a spherical harmonic of positive degree i and of negative degree j is reciprocal. The general (well known) proposition on which this depends is that if V_i is any homogeneous function of (x, y, z) of degree i, positive or negative, integral or fractional, $V_i r^{-2i-1}$ is also a solution of the equation $\nabla^2 V=0$ (see THOMSON and TAIT'S 'Natural Philosophy,' chap. i. Appendix B.).

A spherical harmonic of integral whether positive or negative degree, satisfying the differential equation continuously for all values of the variables, will be called an "entire spherical harmonic," because such functions are suited for the solution of acoustical and other physical problems regarding entire spheres or entire spherical shells.

A spherical harmonic function of (x, y, z) will be called a "spherical surface-harmonic" when the point (x, y, z) lies anywhere on a spherical surface having its centre at the origin of coordinates. A *spherical surface-harmonic* is therefore a function of two variables, angular coordinates of a point on a spherical surface. If Y_i denote such a function of order i, positive and integral, then $Y_i r^i$ and $Y_i r^{-i-1}$ are what we now call simply *spherical harmonics*; but sometimes we shall call them, by way of distinction, "spherical solid harmonics." Functions Y_i, or spherical surface-harmonics of integral orders, have been generally called "LAPLACE'S coefficients" by English writers.

9. From the theorem enunciated in § 7, we see that the general solution of our problem, so far as δ is concerned, is this:—

$$\delta=\Sigma_{i=0}^{i=\infty}(V_i+V'_i r^{-2i-1}). \quad \ldots \ldots \ldots \quad (8)$$

10. Now because the equation $\nabla^2 u=0$ is linear, it follows that differential coefficients of any solution, with reference to x, y, z, or linear functions of such differential coefficients, are also solutions. Hence the terms V_i and $V'_i r^{-2i-1}$, of δ, give harmonics of the degrees $i-1$ and $-(i+2)$, in $\frac{d\delta}{dx}, \frac{d\delta}{dy}, \frac{d\delta}{dz}$. To solve equations (3) we have therefore only to solve

$$\nabla^2 u=\phi_n,$$

where ϕ_n denotes an entire spherical harmonic of any positive or negative degree, n. Trying

$$u=Ar^2\phi_n,$$

which is obviously the right form, we have

$$\nabla^2 u=A\left\{r^2\nabla^2\phi_n+4\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)\phi_n+\phi_n\nabla^2(r^2)\right\}.$$

But, because ϕ_n is a homogeneous function of x, y, z of degree n,

$$\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)\phi_n=n\phi_n;$$

and because it is a spherical harmonic,

$$\nabla^2\phi_n=0.$$

We have also

$$\nabla^2(r^2)=6,$$

by differentiation. Hence

$$\nabla^2 u=A.2(2n+3)\phi_n,$$

and therefore the complete solution of the equation

$$\nabla^2 u=\phi_n$$

is

$$u=V+\frac{r^2}{2(2n+3)}\phi_n,$$

where V denotes any solution of the equation

$$\nabla^2 V=0.$$

11. Hence, by taking for ϕ_n the terms of $\frac{d\delta}{dx}$, $\frac{d\delta}{dy}$, $\frac{d\delta}{dz}$ above referred to (§ 10), and giving n its proper value, $i-1$, or $-(i+2)$, for each term as the case may be, we find, for the complete solution of (3), the following:—

$$\left.\begin{aligned}\alpha&=\Sigma\left\{u_i+u'_i r^{-2i-1}-\frac{mr^2}{n\,.\,2(2i+1)}\frac{d}{dx}(\mathrm{V}_i-\mathrm{V}'_i r^{-2i-1})\right\},\\ \beta&=\Sigma\left\{v_i+v'_i r^{-2i-1}-\frac{mr^2}{n\,.\,2(2i+1)}\frac{d}{dy}(\mathrm{V}_i-\mathrm{V}'_i r^{-2i-1})\right\},\\ \gamma&=\Sigma\left\{w_i+w'_i r^{-2i-1}-\frac{mr^2}{n\,.\,2(2i+1)}\frac{d}{dz}(\mathrm{V}_i-\mathrm{V}'_i r^{-2i-1})\right\},\end{aligned}\right\}\quad\ldots\ldots\quad(9)$$

where u_i, u'_i, v_i, v'_i, w_i, w'_i denote six harmonics, each of degree i.

12. But in order that these formulæ may express the solution of the original equations (1), the functions u, v, &c. must be related to the functions V so as to satisfy equations (2) and (3). Now, taking account of the following formula,

$$\frac{d}{dx}\left(r^2\frac{d\phi_i}{dx}\right)+\frac{d}{dy}\left(r^2\frac{d\phi_i}{dy}\right)+\frac{d}{dz}\left(r^2\frac{d\phi_i}{dz}\right)=2\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)\phi_i+r^2\nabla^2\phi_i\,,$$

which becomes simply

$$2i\phi_i\,,$$

if ϕ_i is a spherical harmonic of any degree i (whether positive or negative, integral or fractional), we derive from (9) by differentiation, and selection of terms of order i, and of order inverse i (or degree $-i-1$),

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=\Sigma\left\{\psi_i+\psi'_i r^{-2i-1}-\frac{m}{n(2i+1)}[i\mathrm{V}_i+(i+1)\mathrm{V}'_i r^{-2i-1}]\right\},$$

where, for brevity, we put

$$\left.\begin{aligned}\psi_i&=\frac{du_{i+1}}{dx}+\frac{dv_{i+1}}{dy}+\frac{dw_{i+1}}{dz}\,,\\ \text{and}\qquad \psi'_i r^{-2i-1}&=\frac{d(u'_{i-1}r^{-2i+1})}{dx}+\frac{d(v'_{i-1}r^{-2i+1})}{dy}+\frac{d(w'_{i-1}r^{-2i+1})}{dz}\,.\end{aligned}\right\}\quad\ldots\ldots\quad(10)$$

Hence, to satisfy (2) and (8),

$$\mathrm{V}_i=\psi_i-\frac{mi}{n(2i+1)}\mathrm{V}_i$$

and

$$\mathrm{V}'_i=\psi'_i-\frac{m(i+1)}{n(2i+1)}\mathrm{V}'_i\,,$$

from which we find

$$\left.\begin{aligned}\mathrm{V}_i&=\frac{n(2i+1)}{(2n+m)i+n}\psi_i\,,\\ \mathrm{V}'_i&=\frac{n(2i+1)}{(2n+m)i+n+m}\psi'_i\,.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(11)$$

13. Using these in (9), we conclude

$$\left.\begin{aligned}\alpha&=\Sigma_{i=0}^{i=\infty}{}^{*}\left\{u_{i+1}+u'_{i-1}r^{-2i+1}-\frac{mr^2}{2}\frac{d}{dx}\left[\frac{\psi_i}{(2n+m)i+n}-\frac{\psi'_i r^{-2i-1}}{(2n+m)i+n+m}\right]\right\},\\ \beta&=\Sigma_{i=0}^{i=\infty}\left\{v_{i+1}+v'_{i-1}r^{-2i+1}-\frac{mr^2}{2}\frac{d}{dy}\left[\frac{\psi_i}{(2n+m)i+n}-\frac{\psi'_i r^{-2i-1}}{(2n+m)i+n+m}\right]\right\},\\ \gamma&=\Sigma_{i=0}^{i=\infty}\left\{w_{i+1}+w'_{i-1}r^{-2i+1}-\frac{mr^2}{2}\frac{d}{dz}\left[\frac{\psi_i}{(2n+m)i+n}-\frac{\psi'_i r^{-2i-1}}{(2n+m)i+n+m}\right]\right\}\end{aligned}\right\} \quad . \quad (12)$$

for a complete solution of the general equations (1), the equations of equilibrium of an isotropic elastic solid. The circumstances for which this solution is appropriate will be understood when the general proposition of § 7 is duly considered.

14. It remains to show how the harmonics u_i, v_i, w_i, u'_i, v'_i, w'_i are to be determined so as to satisfy the superficial conditions. Let us first suppose these to be that the displacement of every point of the bounding surface is given arbitrarily. Let ΣA_i, ΣB_i, ΣC_i be the harmonic series†, expressing the three components of the displacement at any point of the outer surface of the shell, and $\Sigma A'_i$, $\Sigma B'_i$, $\Sigma C'_i$ the corresponding expressions for the given condition of the inner surface. Thus the surface-equations of condition to be fulfilled are

$$\left.\begin{aligned} r=a&\begin{cases}\alpha=\Sigma A_i,\\ \beta=\Sigma B_i,\\ \gamma=\Sigma C_i,\end{cases}\\ r=a'&\begin{cases}\alpha=\Sigma A'_i,\\ \beta=\Sigma B'_i,\\ \gamma=\Sigma C'_i,\end{cases}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where a and a' denote the radii of the outer and inner surfaces respectively, and A_i, B_i, C_i, A'_i, B'_i, C'_i spherical surface-harmonics of the order i.

15. Now collecting from the series (12) of § 13, which constitute the general expressions for α, β, γ, those terms which, being either solid spherical harmonics of degrees i and $-i-1$, or such functions multiplied by r^2, give, at the boundary, surface-harmonics of the order i, and equating the terms of this order on the two sides of equations (13), we have

* For the case $i=0$, the terms u'_{i-1}, v'_{i-1}, w'_{i-1} may be omitted; but their full interpretation would be to express a displacement without deformation. Thus u'_{-1}, being of degree -1, cannot but be $\frac{A}{r}$, where A is a constant; and therefore $u'_{i-1}r^{-2i+1}$ becomes A when $i=0$.

† That is, series of terms each of which is a spherical surface-harmonic of integral order i. That any function; arbitrarily given over an entire spherical surface, may be so expressed, is a well-known theorem. A demonstration of it is given in THOMSON and TAIT's 'Natural Philosophy,' chap. i. Appendix B, § s.

$$\left.\begin{aligned} u_i+u_i'r^{-2i-1}-\frac{mr^2}{2}\frac{d}{dx}\left[\frac{\psi_{i+1}}{(2n+m)i+3n+m}-\frac{\psi_{i-1}'r^{-2i+1}}{(2n+m)i-n}\right]\begin{cases}=\mathrm{A}_i \text{ when } r=a,\\ =\mathrm{A}_i' \text{ when } r=a',\end{cases}\\ v_i+v_i'r^{-2i-1}-\frac{mr^2}{2}\frac{d}{dy}\left[\frac{\psi_{i+1}}{(2n+m)i+3n+m}-\frac{\psi_{i-1}'r^{-2i+1}}{(2n+m)i-n}\right]\begin{cases}=\mathrm{B}_i \text{ when } r=a,\\ =\mathrm{B}_i' \text{ when } r=a',\end{cases}\\ w_i+w_i'r^{-2i-1}-\frac{mr^2}{2}\frac{d}{dz}\left[\frac{\psi_{i+1}}{(2n+m)i+3n+m}-\frac{\psi_{i-1}'r^{-2i+1}}{(2n+m)i-n}\right]\begin{cases}=\mathrm{C}_i \text{ when } r=a,\\ =\mathrm{C}_i' \text{ when } r=a'.\end{cases} \end{aligned}\right\} \quad . \quad . \quad (15)$$

16. These six equations would suffice to determine the six harmonics u_i, v_i, w_i, u_i', v_i', w_i', if ψ_{i+1} and ψ_{i-1}' were known. For, since each of those six functions is a homogeneous function of x, y, z of order i, each of them divided by r^i is a function of angular coordinates relative to the centre, and independent of r; and therefore if, for instance, we denote u_i by $r^i\varpi$ and u_i' by $r^i\varpi'$, we have two unknown quantities ϖ and ϖ' to be determined by the two equations of condition relative to α for the outer and the inner surface. These equations may be written as follows, if we further denote $\frac{d\psi_{i+1}}{dx}$ by $r^i\vartheta$, and $\frac{d(\psi_{i-1}'r^{-2i+1})}{dx}$ by $r^{-i-1}\vartheta'$, because these are homogeneous functions of the orders i and $-i-1$ respectively:

$$\varpi a^{2i+1}+\varpi'=\mathrm{A}_i a^{i+1}+\frac{ma^{2i+3}}{2[(2n+m)i+3n+m]}\vartheta-\frac{ma^2}{2[(2n+m)i-n]}\vartheta',$$

$$\varpi a'^{2i+1}+\varpi'=\mathrm{A}_i' a'^{i+1}+\frac{ma'^{2i+3}}{2[(2n+m)i+3n+m]}\vartheta-\frac{ma'^2}{2[(2n+m)i-n]}\vartheta'.$$

Resolving these equations for ϖ and ϖ', and returning to the original notation instead of ϖ, ϖ', ϑ, ϑ',

$$u_i=\frac{(a^{i+1}\mathrm{A}_i-a'^{i+1}\mathrm{A}_i')r^i+(a^{2i+3}-a'^{2i+3})\mathrm{M}_{i+2}\frac{d\psi_{i+1}}{dx}-(a^2-a'^2)\mathrm{M}_{i-2}'\frac{d(\psi_{i-1}'r^{-2i+1})}{dx}r^{2i+1}}{a^{2i+1}-a'^{2i+1}},$$

$$u_i'=\frac{(aa')^{i+1}(a^i\mathrm{A}_i'-a'^i\mathrm{A}_i)r^i-(aa')^{2i+1}(a^2-a'^2)\mathrm{M}_{i+2}\frac{d\psi_{i+1}}{dx}-(aa')^2(a^{2i-1}-a'^{2i-1})\mathrm{M}_{i-2}'\frac{d(\psi_{i-1}'r^{-2i+1})}{dx}r^{2i+1}}{a^{2i+1}-a'^{2i+1}},$$

where, for brevity,

$$\left.\begin{aligned} \mathrm{M}_{i+2}&=\frac{m}{2[(2n+m)i+3n+m]},\\ \mathrm{M}_{i-2}'&=\frac{m}{2[(2n+m)i-n]}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Introducing, also for brevity, the following notation,

$$\left.\begin{aligned} \mathfrak{A}_i&=\frac{a^{i+1}\mathrm{A}_i-a'^{i+1}\mathrm{A}_i'}{a^{2i+1}-a'^{2i+1}},\\ \mathfrak{A}_i'&=\frac{(aa')^{i+1}(a^i\mathrm{A}_i'-a'^i\mathrm{A}_i)}{a^{2i+1}-a'^{2i+1}}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\left.\begin{aligned} \mathfrak{M}_{i+2}&=\frac{a^{2i+3}-a'^{2i+3}}{a^{2i+1}-a'^{2i+1}}\mathrm{M}_{i+2}, & \mathfrak{M}'_{i-2}&=\frac{a^2-a'^2}{a^{2i+1}-a'^{2i+1}}\mathrm{M}'_{i-2},\\ \mathfrak{N}_{i+2}&=\frac{(aa')^{2i+1}(a^2-a'^2)}{a^{2i+1}-a'^{2i+1}}\mathrm{M}_{i+2}, & \mathfrak{N}_{i-2}'&=\frac{(aa')^2(a^{2i-1}-a'^{2i-1})}{a^{2i+1}-a'^{2i+1}}\mathrm{M}_{i-2}', \end{aligned}\right\} \quad . \quad . \quad . \quad (18)$$

we have the expressions for u_i and u'_i given below. Dealing with the equations of condition relative to β and γ, and introducing an abbreviated notation B_i, B'_i, C_i, C'_i corresponding to (17), we find similar expressions for v_i, v'_i, w_i, w'_i, as follows:—

$$\left.\begin{aligned} u_i &= \mathfrak{A}_i r^i + \mathfrak{M}_{i+2}\frac{d\psi_{i+1}}{dx} - \mathfrak{M}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dx}r^{2i+1}, \\ v_i &= \mathfrak{B}_i r^i + \mathfrak{M}_{i+2}\frac{d\psi_{i+1}}{dy} - \mathfrak{M}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dy}r^{2i+1}, \\ w_i &= \mathfrak{C}_i r^i + \mathfrak{M}_{i+2}\frac{d\psi_{i+1}}{dz} - \mathfrak{M}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dz}r^{2i+1}, \end{aligned}\right\} \quad \ldots \ldots \quad (19)$$

$$\left.\begin{aligned} u'_i &= \mathfrak{A}'_i r^i - \mathfrak{N}_{i+2}\frac{d\psi_{i+1}}{dx} - \mathfrak{N}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dx}r^{2i+1}, \\ v'_i &= \mathfrak{B}'_i r^i - \mathfrak{N}_{i+2}\frac{d\psi_{i+1}}{dy} - \mathfrak{N}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dy}r^{2i+1}, \\ w'_i &= \mathfrak{C}'_i r^i - \mathfrak{N}_{i+2}\frac{d\psi_{i+1}}{dz} - \mathfrak{N}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dz}r^{2i+1}. \end{aligned}\right\} \quad \ldots \ldots \quad (20)$$

17. It only remains to determine the functions ψ and ψ', which we can do by combining these last equations with (10) of § 12. Thus, changing i into $i+1$ in (17) and into $i-1$ in (18), applying equations (10) of § 12, and taking advantage of the following properties,

$$\nabla^2\psi_{i+2}=0,\ \nabla^2(\psi'_i r^{-2i-1})=0,\ \&c.,$$

$$x\frac{d(\psi'_i r^{-2i-1})}{dx}+y\frac{d(\psi'_i r^{-2i-1})}{dy}+z\frac{d(\psi'_i r^{-2i-1})}{dz}=-(i+1)\psi'_i,$$

and

$$x\frac{d\psi_i}{dx}+y\frac{d\psi_i}{dy}+z\frac{d\psi_i}{dz}=i\psi_i,$$

we find

$$\left.\begin{aligned} \psi_i &= \frac{d(\mathfrak{A}_{i+1}r^{i+1})}{dx}+\frac{d(\mathfrak{B}_{i+1}r^{i+1})}{dy}+\frac{d(\mathfrak{C}_{i+1}r^{i+1})}{dz}+(2i+3)(i+1)\mathfrak{M}'_{i-1}\psi'_i, \\ \psi'_i &= \left\{\frac{d(\mathfrak{A}'_{i-1}r^{-i})}{dx}+\frac{d(\mathfrak{B}'_{i-1}r^{-i})}{dy}+\frac{d(\mathfrak{C}'_{i-1}r^{-i})}{dz}\right\}r^{2i+1}+(2i-1)i\mathfrak{N}_{i+1}\psi_i. \end{aligned}\right\} \quad \ldots \quad (21)$$

These equations, used to determine the two unknown functions ψ_i and ψ'_i, give

$$\left.\begin{aligned} \psi_i &= \frac{\Theta_i+(2i+3)(i+1)\mathfrak{M}'_{i+1}\Theta'_i}{1-(2i+3)(2i-1)(i+1)i\mathfrak{M}_{i+1}\mathfrak{N}_{i+1}}, \\ \psi'_i &= \frac{(2i-1)i\mathfrak{N}_{i+1}\Theta_i+\Theta'_i}{1-(2i+3)(2i-1)(i+1)i\mathfrak{M}'_{i+1}\mathfrak{N}_{i+1}}, \end{aligned}\right\} \quad \ldots \ldots \quad (22)$$

where, for brevity,

$$\left.\begin{aligned} \Theta_i &= \frac{d(\mathfrak{A}_{i+1}r^{i+1})}{dx}+\frac{d(\mathfrak{B}_{i+1}r^{i+1})}{dy}+\frac{d(\mathfrak{C}_{i+1}r^{i+1})}{dz}, \\ \Theta'_i &= \left\{\frac{d(\mathfrak{A}'_{i-1}r^{-i})}{dx}+\frac{d(\mathfrak{B}'_{i-1}r^{-i})}{dy}+\frac{d(\mathfrak{C}'_{i-1}r^{-i})}{dz}\right\}r^{2i+1}. \end{aligned}\right\} \quad \ldots \ldots \quad (23)$$

18. The functions ψ and ψ' being expressed in terms of the data of the problem by equations (22), (23), (17), (18), (16), we have only to use (19) and (20) in (12) to find the following expression of the complete solution:—

$$\left.\begin{aligned} \alpha&=\Sigma\left\{\mathfrak{A}_i r^i+\mathfrak{A}'_i r^{-i-1}+(\mathfrak{M}_i-\mathfrak{N}_i r^{-2i+3}-\mathrm{M}_i r^2)\frac{d\psi_{i-1}}{dx}-(\mathfrak{M}'_i r^{2i+5}+\mathfrak{N}'_i-\mathrm{M}'_i r^2)\frac{d(\psi'_{i+1}r^{-2i-3})}{dx}\right\},\\ \beta&=\Sigma\left\{\mathfrak{B}_i r^i+\mathfrak{B}'_i r^{-i-1}+(\mathfrak{M}_i-\mathfrak{N}_i r^{-2i+3}-\mathrm{M}_i r^2)\frac{d\psi_{i-1}}{dy}-(\mathfrak{M}'_i r^{2i+5}+\mathfrak{N}'_i-\mathrm{M}'_i r^2)\frac{d(\psi'_{i+1}r^{-2i-3})}{dy}\right\},\\ \gamma&=\Sigma\left\{\mathfrak{C}_i r^i+\mathfrak{C}'_i r^{-i-1}+(\mathfrak{M}_i-\mathfrak{N}_i r^{-2i+3}-\mathrm{M}_i r^2)\frac{d\psi_{i-1}}{dz}-(\mathfrak{M}'_i r^{2i+5}+\mathfrak{N}'_i-\mathrm{M}'_i r^2)\frac{d(\psi'_{i+1}r^{-2i-3})}{dz}\right\}. \end{aligned}\right\} \quad (24)$$

19. This solution leads immediately, through an extreme case of its application, to the solution of the general problem for a plate of elastic substance between two infinite parallel planes:—Given the displacement of every point of its surface, required the displacement of any interior point. For if we give infinite values to a and a', and keep $a-a'$ finite, the spherical shell becomes an infinite plane plate.

20. It is, however, less easy to deduce the result in this way from the solution for the spherical shell, than to apply directly the general method of § 6 to the case of the infinite plane plate. We shall return to this subject (§ 31, below), when the details of the investigation will be sufficiently indicated.

21. A very important part of the general problem proposed in § 1 remains to be considered,—that in which not the displacement, but the arbitrarily applied force, is given all over the surface. To express the surface-equations of condition for such data, we must use the formulæ expressing the stress (or force of elasticity) in any part of an elastic solid in terms of the strain (or deformation) of the substance. These are

$$\left.\begin{aligned} \mathrm{P}&=(m+n)\frac{d\alpha}{dx}+(m-n)\left(\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right);\\ \mathrm{Q}&=(m+n)\frac{d\beta}{dy}+(m-n)\left(\frac{d\gamma}{dz}+\frac{d\alpha}{dx}\right);\\ \mathrm{R}&=(m+n)\frac{d\gamma}{dz}+(m-n)\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}\right);\\ \mathrm{S}&=n\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right);\ \mathrm{T}=n\left(\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right);\ \mathrm{U}=n\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right); \end{aligned}\right\} \quad . \ . \ . \ . \ (25)$$

where P, Q, R are the normal tractions (which when negative are pressures) on the faces of a unit cube respectively perpendicular to the lines of reference OX, OY, OZ, and S, T, U the tangential forces along the faces respectively parallel, and in the directions in these planes respectively perpendicular, to OX, OY, OZ (see Appendix, § 70).

22. In terms of these we have the following expressions for the components F, G, H of the force on a unit area perpendicular to any line whose direction cosines are f, g, h:—

$$\left.\begin{aligned} \mathrm{F}&=\mathrm{P}f+\mathrm{U}g+\mathrm{T}h,\\ \mathrm{G}&=\mathrm{U}f+\mathrm{Q}g+\mathrm{S}h,\\ \mathrm{H}&=\mathrm{T}f+\mathrm{S}g+\mathrm{R}h \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (26)$$

(see "Elements of a Mathematical Theory of Elasticity," Philosophical Transactions for 1856, p. 481).

23. Using the expressions (25) in (26), we find

$$F=(m-n)\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)f+n\left(\frac{d\alpha}{dx}f+\frac{d\alpha}{dy}g+\frac{d\alpha}{dz}h\right)+n\left(\frac{d\alpha}{dx}f+\frac{d\beta}{dx}g+\frac{d\gamma}{dx}h\right), \quad . \quad (27)$$

and symmetrical expressions for G and H.

24. If now we suppose f, g, h to denote the direction-cosines of the normal at any point x, y, z of the surface of an elastic solid, the surface condition, when force, not displacement, is given, will be expressed by equating F, G, H respectively to three functions of the coordinates of a point in the surface, quite arbitrary except in so far as they must balance one another in order that equilibrium in the body may be possible; and therefore they must fulfil the following integral equations:—

$$\iint Fd\Omega=0,\ \iint Gd\Omega=0,\ \iint Hd\Omega=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\iint(Hy-Gz)d\Omega=0,\ \iint(Fz-Hx)d\Omega=0,\ \iint(Gx-Fy)=0, \quad . \quad . \quad . \quad (29)$$

where $d\Omega$ denotes an element of the surface at the point (x, y, z), and the double integrals include the whole surface of application of the forces F, G, H.

25. For our case of the spherical shell, with origin of coordinates at its centre, we have

$$f=\frac{x}{r},\ g=\frac{y}{r},\ h=\frac{z}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and the last triple term in the expression (27) for F may be conveniently written thus:—

$$\frac{n}{r}\frac{d(\alpha x+\beta y+\gamma z)}{dx}-\frac{n\alpha}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Then, for brevity, putting

$$\alpha x+\beta y+\gamma z=\zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and

$$x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}=r\frac{\mathrm{d}}{dr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where $\frac{\mathrm{d}}{dr}$ prefixed to any function of x, y, z will denote its rate of variation per unit of length in the radial direction; and using (2) of § 3, we have, by (30) and the symmetrical equations for G and H,

$$\left.\begin{aligned} Fr&=(m-n)\delta.x+n\left\{\left(r\frac{\mathrm{d}}{dr}-1\right)\alpha+\frac{d\zeta}{dx}\right\},\\ Gr&=(m-n)\delta.y+n\left\{\left(r\frac{\mathrm{d}}{dr}-1\right)\beta+\frac{d\zeta}{dy}\right\},\\ Hr&=(m-n)\delta.z+n\left\{\left(r\frac{\mathrm{d}}{dr}-1\right)\gamma+\frac{d\zeta}{dz}\right\}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

26. It is to be remarked that these equations express such functions of (x, y, z), the coordinates of any point P of the solid, that $F.\omega$, $G.\omega$, $H.\omega$ are the three components of

the force transmitted across an infinitely small area ω perpendicular to OP, while, for any point of either the outer or the inner bounding spherical surface, $F\omega$, $G\omega$, $H\omega$ are the three components of the force applied to an infinitely small element ω of this surface.

27. To reduce the surface-equations of condition derived from these expressions to harmonic equations, let us consider homogeneous terms of degree i of the complete solution, which we shall denote by α_i, β_i, γ_i, and let δ_{i-1}*, ζ_{i+1} denote the corresponding terms of the other functions. Thus we have

$$\left.\begin{aligned} Fr &= \Sigma\left\{(m-n)\delta_{i-1}x+n(i-1)\alpha_i+n\frac{d\zeta_{i+1}}{dx}\right\}, \\ Gr &= \Sigma\left\{(m-n)\delta_{i-1}y+n(i-1)\beta_i+n\frac{d\zeta_{i+1}}{dy}\right\}, \\ Hr &= \Sigma\left\{(m-n)\delta_{i-1}z+n(i-1)\gamma_i+n\frac{d\zeta_{i+1}}{dz}\right\}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

28. The second of the three terms of order i in these equations, when the general solution of § 13 is used, become at the boundary each explicitly the sum of two surface harmonics of orders i and $i-2$ respectively. To bring the other parts of the expressions to similar forms, it is convenient that we should first express ζ_{i+1} in terms of the general solution (12) of § 13, by selecting the terms of algebraic degree i. Thus we have

$$\alpha_i=u_i-\frac{mr^2}{2[(2n+m)i-n-m]}\frac{d\psi_{i-1}}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and symmetrical expressions for β_i and γ_i, from which we find

$$\alpha_i x+\beta_i y+\gamma_i z=\zeta_{i+1}=u_i x+v_i y+w_i z-\frac{(i-1)mr^2\psi_{i-1}}{2[(2n+m)i-n-m]}.$$

Hence, by the proper formulæ [see (42) below] for reduction to harmonics,

$$\zeta_{i+1}=-\frac{1}{2i+1}\left\{\frac{(2i-1)[m(i-1)-2n]}{2[(2n+m)i-n-m]}r^2\psi_{i-1}+\phi_{i+1}\right\}, \quad . \quad . \quad . \quad . \quad (37)$$

where

$$\phi_{i+1}=r^{2i+3}\left\{\frac{d(u_i r^{-2i-1})}{dx}+\frac{d(v_i r^{-2i-1})}{dy}+\frac{d(w_i r^{-2i-1})}{dz}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (38)$$

and (as before assumed in § 12)

$$\psi_{i-1}=\frac{du_i}{dx}+\frac{dv_i}{dy}+\frac{dw_i}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Also, by (11) of § 12, or directly from (36) by differentiation, we have

$$\delta_{i-1}=\frac{n(2i-1)}{(2n+m)i-n-m}\cdot\psi_{i-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Substituting these expressions for δ_{i-1}, α_i, and ζ_{i+1} in (35), we find

* When $i-1$ is positive, δ_{i-1} will express the same function as V_{i-1} of § 9 above. The suffixes now introduced have reference solely to the algebraic degree, positive or negative, of the functions, whether harmonic or not, to the symbols for which they are applied.

$$\mathrm{F}r=\Sigma\left\{n(i-1)u_i+\frac{n(2i-1)[(m-2n)i+2m+n]}{(2i+1)[(m+2n)i-m-n]}x\psi_{i-1}-\frac{n[2i(i-1)m-(2i-1)n]}{(2i+1)[(m+2n)i-m-n]}r^2\frac{d\psi_{i-1}}{dx}-\frac{n}{2i+1}\frac{d\phi_{i+1}}{dx}\right\}. \quad (41)$$

This is reduced to the required harmonic form by the obviously proper formula

$$x\psi_{i-1}=\frac{1}{2i-1}\left\{r^2\frac{d\psi_{i-1}}{dx}-r^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}\right\} \quad . \quad . \quad . \quad (42)$$

Thus, and dealing similarly with the expressions for Gr and Hr, we have, finally,

$$\left.\begin{aligned}\mathrm{F}r&=n\Sigma\left\{(i-1)u_i-2(i-2)\mathrm{M}_ir^2\frac{d\psi_{i-1}}{dx}-\mathrm{E}_ir^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\frac{1}{2i+1}\frac{d\phi_{i+1}}{dx}\right\},\\ \mathrm{G}r&=n\Sigma\left\{(i-1)v_i-2(i-2)\mathrm{M}_ir^2\frac{d\psi_{i-1}}{dy}-\mathrm{E}_ir^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\frac{1}{2i+1}\frac{d\phi_{i+1}}{dy}\right\},\\ \mathrm{H}r&=n\Sigma\left\{(i-1)w_i-2(i-2)\mathrm{M}_ir^2\frac{d\psi_{i-1}}{dz}-\mathrm{E}_ir^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\frac{1}{2i+1}\frac{d\phi_{i+1}}{dz}\right\},\end{aligned}\right\} \quad . \quad . \quad (43)$$

where [as above, (16) of § 16.]

and now further

$$\left.\begin{aligned}\mathrm{M}_i&=\frac{1}{2}\,\frac{m}{(m+2n)i-m-n},\\ \mathrm{E}_i&=\frac{(m-2n)i+2m+n}{(2i+1)[(m+2n)i-m-n]}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (44)$$

29. To express the surface conditions by harmonic equations, let us suppose the superficial values of F, G, H to be given as follows:

$$\left.\begin{aligned}&\left.\begin{aligned}\mathrm{F}&=\Sigma\mathrm{A}_i,\\ \mathrm{G}&=\Sigma\mathrm{B}_i,\\ \mathrm{H}&=\Sigma\mathrm{C}_i,\end{aligned}\right\}\text{when } r=a,\\ &\text{and}\\ &\left.\begin{aligned}\mathrm{F}&=\Sigma\mathrm{A}'_i,\\ \mathrm{G}&=\Sigma\mathrm{B}'_i,\\ \mathrm{H}&=\Sigma\mathrm{C}'_i,\end{aligned}\right\}\text{when } r=a',\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (45)$$

where A_i, B_i, C_i, A'_i, B'_i, C'_i denote surface harmonics of order i. Now the terms of algebraic degree i, exhibited in the preceding expressions (43) for Fr, Gr, Hr, become, at either of the concentric spherical surfaces, sums of surface harmonics of orders i and $i-2$, when i is positive, and of orders $-i-1$ and $-i-3$ when i is negative. Hence, selecting all the terms which lead to surface harmonics of order i, and equating to the proper terms of the data (45), we have

$$\frac{n}{r}\left\{\begin{aligned}&(i-1)u_i-(i+2)u_{-i-1}-2i\mathrm{M}_{i+2}r^2\frac{d\psi_{i+1}}{dx}+2(i+1)\mathrm{M}_{-i+1}r^2\frac{d\psi_{-i}}{dx}\\ &-\mathrm{E}_ir^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\mathrm{E}_{-i-1}r^{-2i-1}\frac{d(\psi_{-i-2}r^{2i+3})}{dx}-\frac{1}{2i+1}\left(\frac{d\phi_{i+1}}{dx}-\frac{d\phi_{-i}}{dx}\right)\end{aligned}\right\}=\left\{\begin{aligned}&\mathrm{A}_i\text{ when } r=a,\\ &\mathrm{A}'_i\text{ when } r=a',\end{aligned}\right\} \quad (46)$$

and symmetrical equations relative to y and z.

30 These equations might be dealt with exactly as formerly with the equations (15)

of § 15. But the following order of proceeding is more convenient. Commencing with the first of the surface equations (46), multiplying it by $\left(\frac{r}{a}\right)^i$, attending to the degree of each term, and taking advantage of the principle that, if ψ be any homogeneous function of x, y, z, of degree t, the function of angular coordinates, or of the ratios $x:y:z$, which it becomes at the spherical surface $r=a$, is the same as $\left(\frac{a}{r}\right)^t \psi$ for any value of r, we have

$$\frac{n}{a}\left\{\begin{array}{l}(i-1)u_i-(i+2)\left(\frac{r}{a}\right)^{2i+1}u_{-i-1}-2i\mathrm{M}_{i+2}a^2\frac{d\psi_{i+1}}{dx}+2(i+1)\mathrm{M}_{-i+1}a^2\left(\frac{r}{a}\right)^{2i+1}\frac{d\psi_{-i}}{dx}\\ -\mathrm{E}_i r^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\mathrm{E}_{-i-1}a^{-2i-1}\frac{d(\psi_{-i-2}r^{2i+3})}{dx}-\frac{1}{2i+1}\left[\frac{d\phi_{i+1}}{dx}-\left(\frac{r}{a}\right)^{2i+1}\frac{d\phi_{-i}}{dx}\right]\end{array}\right\}=\mathrm{A}_i\left(\frac{r}{a}\right)^i, \quad (47)$$

where the second member, and each term of the first member, is now a homogeneous function of degree i, of x, y, z (being in fact a solid spherical harmonic of degree and order i). Taking $\frac{d}{dx}$ of this, and $\frac{d}{dy}$ and $\frac{d}{dz}$ of the two symmetrical equations, adding, taking into account equations (38) and (39), and taking advantage of the equation $\nabla^2\mathrm{V}=0$ for the solid harmonic functions concerned, we have

$$\left.\begin{array}{l}\frac{n}{a}\left\{[i-1+(2i+1)i\mathrm{E}_i]\psi_{i-1}-2(i+1)a^{-2i-1}r^{2i-1}\phi_{-i}-2(i+1)(2i+1)i\mathrm{M}_{-i+1}\left(\frac{r}{a}\right)^{2i+1}\psi_{-i}\right\}\\ \qquad=\frac{1}{a^i}\left\{\frac{d(\mathrm{A}_i r^i)}{dx}+\frac{d(\mathrm{B}_i r^i)}{dy}+\frac{d(\mathrm{C}_i r^i)}{dz}\right\}.\end{array}\right\} \quad (48)$$

Again, multiplying (47) by $a^{-2}r^{-2i-1}$, and taking $r^{2i+3}\frac{d}{dx}$ of the result, dealing similarly with the two symmetrical equations, and adding, we have

$$\left.\begin{array}{l}\frac{n}{a}\left\{2ia^{-2}\phi_{i+1}-[i+2-(2i+1)(i+1)\mathrm{E}_{-i-1}]\left(\frac{r}{a}\right)^{2i+3}\psi_{-i-2}+2i(i+1)(2i+1)\mathrm{M}_{i+2}\psi_{i+1}\right\}\\ \qquad=\frac{r^{2i+3}}{a^{i+2}}\left\{\frac{d(\mathrm{A}_i r^{-i-1})}{dx}+\frac{d(\mathrm{B}_i r^{-i-1})}{dy}+\frac{d(\mathrm{C}_i r^{-i-1})}{dz}\right\}.\end{array}\right\} \quad (49)$$

Changing i into $i-2$ in this equation, we have

$$\left.\begin{array}{l}\frac{n}{a}\left\{2(i-2)a^{-2}\phi_{i-1}-[i-(2i-3)(i-1)\mathrm{E}_{-i+1}]\left(\frac{r}{a}\right)^{2i-1}\psi_{-i}+2(i-2)(i-1)(2i-3)\mathrm{M}_i\psi_{i-1}\right\}\\ \qquad=\frac{r^{2i-1}}{a^i}\left\{\frac{d(\mathrm{A}_{i-2}r^{-i+1})}{dx}+\frac{d(\mathrm{B}_{i-2}r^{-i+1})}{dy}+\frac{d(\mathrm{C}_{i-2}r^{-i+1})}{dz}\right\}.\end{array}\right\} \quad (50)$$

Precisely similar equations, derived from the inner surface condition of the shell, are obtained by changing a, A, B, C into a', A′, B′, C′. We thus have (48), (50), and the two corresponding equations for the inner surface, in all four equations, to determine the four unknown functions ψ_{i-1}, ψ_{-i}, ϕ_{i-1}, ϕ_{-i}, in terms of the data which appear in the second members. The equations being simple algebraic equations, we may regard these four functions as explicitly determined. In other words, we may suppose ϕ_i and ψ_i known for every positive or negative integral value of i. Then equation (47), the two equations symmetrical with it, and the others got by changing A, a, &c. into A$'$, a', &c., give u_i, v_i, w_i explicitly in terms of known functions, and the expressions (36) for α_i, β_i, γ_i complete the solution of the problem.

31. The solution for the infinite plane plate is of course included in the general solution for the spherical shell, as remarked above for the case in which surface displacements, not surface forces, were given; but, as in that case, it will be simpler and practically easier to work out the problem *ab initio*, taking advantage of the appropriate Fourier forms. The relative ease of the independent investigation is indeed still greater in the case in which the surface forces are given than in the other case, since the general expressions for the surface forces assume simple forms when the surface is plane, and require no such transformation as that which we have found necessary, and which has constituted the special difficulty of the problem, when the surface was spherical. The problem of the plane plate presents many questions of remarkable interest and practical importance; and although the object and limits of the present paper preclude any detailed investigation of special cases, we may make a short digression to work out the general solution.

32. Let the origin of coordinates be taken in one side of the plate and the axis OX perpendicular to it. Then, according to the general expressions (25) of § 21, the three components of the force per unit of area, in or parallel to either side of the plate, are respectively

$$\left.\begin{aligned}&\text{parallel to OX, } \mathrm{P}=(m+n)\frac{d\alpha}{dx}+(m-n)\left(\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right),\\ &\text{parallel to OY, } \mathrm{U}=n\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right),\\ &\text{parallel to OZ, } \mathrm{T}=n\left(\frac{d\alpha}{dz}+\frac{d\gamma}{dx}\right).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (51)$$

The surface condition to be fulfilled is that each of these functions shall have an arbitrarily given value at every point of each infinite plane side of the plate.

33. From the indications of § 6 above, it is easily seen that the following assumptions are correct for a general solution of the equations of internal equilibrium, and convenient for the application at present proposed,

$$\alpha=u+x\frac{d\phi}{dx},$$

$$\beta=v+x\frac{d\phi}{dy},$$

$$\gamma=w+x\frac{d\phi}{dz},$$

where u, v, w, and ϕ denote functions of (x, y, z) which each fulfil the equation $\nabla^2\mathrm{V}=0$. From these, by differentiation, and by taking $\nabla^2\phi=0$ into account, we have

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}+\frac{d\phi}{dx},$$

or

$$\left.\begin{aligned} \delta&=\psi+\frac{d\phi}{dx} \\ \text{if}\qquad \psi&=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (52)$$

and δ be used with the same signification as above (§ 2). Also, by differentiation and application of the equations $\nabla^2 u=0$, $\nabla^2\frac{d\phi}{dx}=0$, we find

$$\nabla^2\alpha=2\frac{d^2\phi}{dx^2},\quad \nabla^2\beta=2\frac{d^2\phi}{dxdy},\quad \nabla^2\gamma=2\frac{d^2\phi}{dxdz}.$$

Hence, to satisfy the general equations of internal equilibrium (3) of § 3, we must have

$$\frac{d\phi}{dx}=-\frac{m}{m+2n}\psi.$$

Hence the general solution becomes

$$\left.\begin{aligned} \alpha&=u-\frac{mx}{m+2n}\psi, \\ \beta&=v-\frac{mx}{m+2n}\frac{d\int\psi dx}{dy}, \\ \gamma&=w-\frac{mx}{m+2n}\frac{d\int\psi dx}{dz}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (53)$$

where u, v, w are any functions whatever which satisfy the general equation $\nabla^2 V=0$, and ψ is given by (52); and where, further, it must be understood that $\int\psi dx$ must be so assigned as to satisfy the equation $\nabla^2 V=0$, which ψ itself satisfies by virtue of (52).

34. The general form of the solution of $\nabla^2 V=0$, convenient for the present application, is clearly

$$\varepsilon^{\pm px}\,{}^{\sin}_{\cos}(sy)\,{}^{\sin}_{\cos}(tz),$$

where p, s, t are three constants subject to the equation

$$p^2=s^2+t^2.$$

If now we suppose, as a particular case, the surface condition to be that

$$\left.\begin{aligned} &\left.\begin{aligned} P&=A\sin(sy)\sin(tz), \\ U&=B\cos(sy)\sin(tz), \\ T&=C\sin(sy)\cos(tz), \end{aligned}\right\}\text{when } x=0, \\ \text{and}\qquad &\left.\begin{aligned} P&=A'\sin(sy)\sin(tz), \\ U&=B'\cos(sy)\sin(tz), \\ T&=C'\sin(sy)\cos(tz), \end{aligned}\right\}\text{when } x=a, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (54)$$

where A, B, C, A′, B′, C′ are six given constants, we must clearly have

$$\left.\begin{aligned} u&=(f\varepsilon^{-px}+f'\varepsilon^{px})\sin(sy)\sin(tz),\\ v&=(g\varepsilon^{-px}+g'\varepsilon^{px})\cos(sy)\sin(tz),\\ w&=(h\varepsilon^{-px}+h'\varepsilon^{px})\sin(sy)\cos(tz),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (55)$$

where f, g, h, f', g', h' are six constants to be determined by six linear equations obtained directly from (54), (51), (53), (52), (55). But, by proper interchanges of sines and cosines, we have in (54) a representation of the general terms of the series or of the definite integrals, representing, according to FOURIER'S principles, the six arbitrary functions, whether periodic or non-periodic, by which P, U, T are given over each of the two infinite plane sides. Hence the solution thus indicated is complete.

35. To complete the theory of the equilibrium of an elastic spheroidal shell, we must now suppose every point of the solid substance to be urged by a given force. The problem thus presented will be reduced to that already solved, by the following simple investigation.

36. Let X, Y, Z be the components of the force per unit of volume on the substance at any point x, y, z. (That is to say, let qX, qY, qZ be the three components of the actual force on a volume q, infinitely small in all its dimensions, enclosing the point (x, y, z). Not to unnecessarily limit the problem, we must suppose X, Y, Z to be each an absolutely arbitrary function of x, y, z.

37. When we remember that x, y, z are the coordinates of the undisturbed position of any point of the substance, and differ by the infinitely small quantities α, β, γ from the actual coordinates of the same point of the substance in the body disturbed by the applied forces, we perceive that $Xdx+Ydy+Zdz$ need not be the differential of a function of three independent variables. It actually will not be a complete differential if the case be that of the interior kinetic equilibrium of a rigid body starting from rest under the influence of given constant forces applied to its surface, and having for their resultant a couple in a plane perpendicular to a principal axis. Nor will $Xdx+Ydy+Zdz$ be a complete differential in the interior of a steel bar-magnet held at rest under the influence of an electric current directed through one half of its length, as we perceive when we consider FARADAY'S beautiful experiment showing rotation to supervene in this case when the magnet is freed from all mechanical constraint.

38. The equations of elastic equilibrium are of course now

$$\left.\begin{aligned} n\nabla^2\alpha+m\frac{d\delta}{dx}&=-X,\\ n\nabla^2\beta+m\frac{d\delta}{dy}&=-Y,\\ n\nabla^2\gamma+m\frac{d\delta}{dz}&=-Z.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (56)$$

Let ϖ, ϱ, σ denote some three particular solutions of the equations

$$\left.\begin{aligned}\nabla^2\varpi&=-\mathrm{X},\\ \nabla^2\varrho&=-\mathrm{Y},\\ \nabla^2\sigma&=-\mathrm{Z}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (57)$$

These, ϖ, ϱ, σ, we may regard as known functions, being derivable from X, Y, Z by known methods (THOMSON and TAIT'S 'Natural Philosophy,' chap. vi.). Then, if we assume

$$\left.\begin{aligned}\alpha-\frac{\varpi}{n}&=\alpha_{\prime},\\ \beta-\frac{\varrho}{n}&=\beta_{\prime},\\ \gamma-\frac{\sigma}{n}&=\gamma_{\prime},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (58)$$

and

$$\frac{d\alpha_{\prime}}{dx}+\frac{d\beta_{\prime}}{dy}+\frac{d\gamma_{\prime}}{dz}=\delta_{\prime}, \quad . \quad . \quad . \quad . \quad (59)$$

the equations (56) of interior equilibrium become

$$\left.\begin{aligned}n\nabla^2\alpha_{\prime}+m\frac{d\delta_{\prime}}{dx}&=-\frac{m}{n}\frac{d\xi}{dx},\\ n\nabla^2\beta_{\prime}+m\frac{d\delta_{\prime}}{dy}&=-\frac{m}{n}\frac{d\xi}{dy},\\ n\nabla^2\gamma_{\prime}+m\frac{d\delta_{\prime}}{dz}&=-\frac{m}{n}\frac{d\xi}{dz},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (60)$$

where ξ is a known function given by the equation

$$\xi=\frac{d\varpi}{dx}+\frac{d\varrho}{dy}+\frac{d\sigma}{dz}. \quad . \quad . \quad . \quad . \quad (61)$$

Now, as we verify in a moment by differentiation, equations (59) and (60) are satisfied by

$$\left.\begin{aligned}\alpha_{\prime}&=\frac{-m}{n(m+n)}\frac{d\vartheta}{dx},\\ \beta_{\prime}&=\frac{-m}{n(m+n)}\frac{d\vartheta}{dy},\\ \gamma_{\prime}&=\frac{-m}{n(m+n)}\frac{d\vartheta}{dz},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (62)$$

if ϑ is some particular solution of

$$\nabla^2\vartheta=\xi. \quad . \quad . \quad . \quad . \quad (63)$$

Hence (58), (57), (62), (63), (61) express a particular solution of (56).

39. We conclude that the general solution of (56) may be expressed thus:—

$$\left.\begin{aligned}\alpha&=\frac{1}{n}\left(\varpi-\frac{m}{m+n}\frac{d\vartheta}{dx}\right)+{}'\alpha,\\ \beta&=\frac{1}{n}\left(\varrho-\frac{m}{m+n}\frac{d\vartheta}{dy}\right)+{}'\beta,\\ \gamma&=\frac{1}{n}\left(\sigma-\frac{m}{m+n}\frac{d\vartheta}{dz}\right)+{}'\gamma,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (64)$$

where

$$\left.\begin{aligned}\varpi&=\nabla^{-2}\mathrm{X},\\ \varrho&=\nabla^{-2}\mathrm{Y},\\ \sigma&=\nabla^{-2}\mathrm{Z},\\ \vartheta&=\nabla^{-2}\left(\frac{d\varpi}{dx}+\frac{d\varrho}{dy}+\frac{d\sigma}{dz}\right),\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (65)$$

according to an abbreviated notation, which explains itself sufficiently; and $'\alpha$, $'\beta$, $'\gamma$ denote a general solution of the equations

$$\left.\begin{aligned}n\nabla^{2\prime}\alpha+m\frac{d}{dx}\left(\frac{d\,'\alpha}{dx}+\frac{d\,'\beta}{dy}+\frac{d\,'\gamma}{dz}\right)=0,\\ n\nabla^{2\prime}\beta+m\frac{d}{dy}\left(\frac{d\,'\alpha}{dx}+\frac{d\,'\beta}{dy}+\frac{d\,'\gamma}{dz}\right)=0,\\ n\nabla^{2\prime}\gamma+m\frac{d}{dz}\left(\frac{d\,'\alpha}{dx}+\frac{d\,'\beta}{dy}+\frac{d\,'\gamma}{dz}\right)=0.\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (66)$$

40. This solution is applicable of course to an elastic body of any shape. It enables us to determine the displacement of every point of it when any given force is applied to every point of its interior, and either displacements or forces are given over the whole surface, if we can solve the general problem for the same shape of body with arbitrary superficial data, but no force on the interior parts. For $'\alpha$, $'\beta$, $'\gamma$ are determined by the solution of this problem, to be worked out with the given arbitrary superficial functions modified by the subtraction from them of terms due to the parts of α, β, γ which are explicitly shown in terms of data by equations (64) and (65).

41. Hence the problem of § 35 is completely solved,—whether we have *displacements* given over each of the two concentric spherical bounding surfaces, when the solution of §§ 14–18 determines $'\alpha$, $'\beta$, $'\gamma$; or *forces* given over the boundary, when the solution of §§ 26–30 is available. In the former case the superficial values of the functions

$$\frac{1}{n}\left(\varpi-\frac{m}{m+n}\frac{d\vartheta}{dx}\right),$$

$$\frac{1}{n}\left(\varrho-\frac{m}{m+n}\frac{d\vartheta}{dy}\right),$$

$$\frac{1}{n}\left(\sigma-\frac{m}{m+n}\frac{d\vartheta}{dz}\right),$$

known from equations (65), must be subtracted from the arbitrary functions given as the superficial values of α, β, γ, and the residues, expressed in surface-harmonic series by the known method, will be the harmonic expressions for the superficial values of $'\alpha$, $'\beta$, $'\gamma$. In the latter case, we must first substitute those known functions $\frac{1}{n}\left(\varpi-\frac{m}{m+n}\frac{d\vartheta}{dx}\right)$, &c., instead of α, β, γ respectively in (34), and the values of $\mathrm{F}r$, $\mathrm{G}r$, $\mathrm{H}r$ thus found must be subtracted from the given arbitrary functions representing the true superficial values of Fr, Gr, Hr. The remainders, which we may denote by 'Fr, 'Gr, 'Hr, must then be reduced to harmonic series, as in (45), and used according to the investigation of § 30, to determine $'\alpha$, $'\beta$, $'\gamma$.

42. The general solution (64) and the expression just indicated for the terms to be subtracted from the data so as to find 'Fr, 'Gr, 'Hr, becomes much simplified when, as in some of the most important practical applications, $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$ is a complete differential. Thus let

$$-\mathrm{X}=\frac{d\mathrm{W}}{dx},\quad -\mathrm{Y}=\frac{d\mathrm{W}}{dy},\quad -\mathrm{Z}=\frac{d\mathrm{W}}{dz}, \quad \ldots\ldots\ldots \quad (67)$$

W denoting any function of x, y, z. Then, assuming, as we may do according to (65),

$$\varpi=\frac{d}{dx}\nabla^{-2}\mathrm{W},$$

$$\varrho=\frac{d}{dy}\nabla^{-2}\mathrm{W},$$

$$\sigma=\frac{d}{dz}\nabla^{-2}\mathrm{W},$$

we have by differentiating, &c.,

$$\frac{d\varpi}{dx}+\frac{d\varrho}{dy}+\frac{d\sigma}{dz}=\mathrm{W},$$

and therefore

$$\vartheta=\nabla^{-2}\mathrm{W}. \quad \ldots\ldots\ldots \quad (68)$$

Hence the solution (64) becomes

$$\left.\begin{aligned}\alpha&=\frac{1}{m+n}\frac{d\vartheta}{dx}+'\alpha,\\ \beta&=\frac{1}{m+n}\frac{d\vartheta}{dy}+'\beta,\\ \gamma&=\frac{1}{m+n}\frac{d\vartheta}{dz}+'\gamma.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (69)$$

From this we find

$$\delta=\frac{1}{m+n}\mathrm{W}+'\delta,$$

and (§ 25)

$$\zeta=\frac{1}{m+n}r\frac{\mathrm{d}\vartheta}{\mathrm{d}r}+'\zeta,$$

if

$$'\delta=\frac{d'\alpha}{dx}+\frac{d'\beta}{dy}+\frac{d'\gamma}{dz},$$

and

$$'\zeta='\alpha x+'\beta y+'\gamma z. \quad \ldots\ldots\ldots \quad (70)$$

Hence, by (34),

$$\mathrm{F}r=\frac{1}{m+n}\left\{(m-n)\mathrm{W}x+n\left[\left(r\frac{\mathrm{d}}{\mathrm{d}r}-1\right)\frac{d}{dx}+\frac{d}{dx}r\frac{\mathrm{d}}{\mathrm{d}r}\right]\vartheta\right\}+'\mathrm{F}r.$$

But

$$\frac{d}{dx}r\frac{\mathrm{d}}{\mathrm{d}r}=\left(r\frac{\mathrm{d}}{\mathrm{d}r}+1\right)\frac{d}{dx}.$$

Thus for Fr, and the symmetrical expressions, we have

$$\left.\begin{aligned}\mathrm{F}r&=\frac{1}{m+n}\left\{(m-n)\mathrm{W}x+2nr\frac{\mathrm{d}}{dr}\frac{d\vartheta}{dx}\right\}+'\mathrm{F}r,\\ \mathrm{G}r&=\frac{1}{m+n}\left\{(m-n)\mathrm{W}y+2nr\frac{\mathrm{d}}{dr}\frac{d\vartheta}{dy}\right\}+'\mathrm{G}r,\\ \mathrm{H}r&=\frac{1}{m+n}\left\{(m-n)\mathrm{W}z+2nr\frac{\mathrm{d}}{dr}\frac{d\vartheta}{dz}\right\}+'\mathrm{H}r.\end{aligned}\right\} \quad \ldots \ldots \quad (71)$$

43. These expressions become further simplified if W is a homogeneous function of any positive or negative integral or fractional order $i+1$, in which case we shall denote it by W_{i+1}. For ϑ will be a homogeneous function of order $i+3$, and $\frac{d\vartheta}{dx}$ of order $i+2$. Hence

$$r\frac{\mathrm{d}}{dr}\frac{d}{dx}\vartheta=(i+2)\frac{d\vartheta}{dx}.$$

Hence the preceding become

$$\left.\begin{aligned}\mathrm{F}r&=\frac{1}{m+n}\left\{(m-n)\mathrm{W}_{i+1}\ x+2n(i+2)\frac{d\vartheta}{dx}\right\}+'\mathrm{F}r,\\ \mathrm{G}r&=\frac{1}{m+n}\left\{(m-n)\mathrm{W}_{i+1}\ y+2n(i+2)\frac{d\vartheta}{dy}\right\}+'\mathrm{G}r,\\ \mathrm{H}r&=\frac{1}{m+n}\left\{(m-n)\mathrm{W}_{i+1}\ z+2n(i+2)\frac{d\vartheta}{dz}\right\}+'\mathrm{H}r.\end{aligned}\right\} \quad \ldots \ldots \quad (72)$$

44. These expressions are the more readily reduced to the harmonic forms proper for working out the solution, if the interior force potential, W_{i+1}, is itself a harmonic function. We then have (§ 10)

$$\left.\begin{aligned}&\vartheta=\frac{1}{2(2i+5)}r^2\mathrm{W}_{i+1},\quad \frac{d\vartheta}{dx}=\frac{1}{2i+5}\left(x\mathrm{W}_{i+1}+\frac{1}{2}r^2\frac{d\mathrm{W}_{i+1}}{dx}\right)\\ \text{and}\quad &\mathrm{W}_{i+1}x=\frac{1}{2i+3}\left\{r^2\frac{d\mathrm{W}_{i+1}}{dx}-r^{2i+5}\frac{d(\mathrm{W}_{i+1}r^{-2i-3})}{dx}\right\},\end{aligned}\right\} \quad \ldots \ldots \quad (73)$$

which give

$$\mathrm{F}r=\frac{1}{m+n}\left\{\frac{m+(i+1)n}{2i+3}r^2\frac{d\mathrm{W}_{i+1}}{dx}-\frac{m(2i+5)-n}{(2i+3)(2i+5)}r^{2i+5}\frac{d(\mathrm{W}_{i+1}r^{-2i-3})}{dx}\right\}+'\mathrm{F}r, \quad \ldots \quad (74)$$

and symmetrical expressions for Gr and Hr. Here the terms to be subtracted from the arbitrary functions given to represent the superficial values of Fr, Gr, and Hr are each explicitly expressed in sums of two surface harmonics of orders i or $-i-1$, and $i+2$ or $-i-3$ respectively, viz., in each case, that one of the two numbers which is not negative.

45. When the shell is in equilibrium under the influence of the forces acting on it through its interior, without any application of force to its surface, we must have

$$\left.\begin{aligned}\mathrm{F}r&=0,\\ \mathrm{G}r&=0,\\ \mathrm{H}r&=0,\end{aligned}\right\}\text{when } r=a \text{ and when } r=a'. \quad \ldots \ldots \quad (75)$$

Hence, for the case in which W is a spherical harmonic, the preceding equations give the proper harmonic expressions for $'\mathrm{F}r$, $'\mathrm{G}r$, $'\mathrm{H}r$ at the outer and inner bounding surfaces, for determining $'\alpha$, $'\beta$, $'\gamma$ by the method of §§ 28–30. Thus, using all the same notations, with the exception of $'\alpha$, $'\beta$, $'\gamma$, $'\mathrm{F}$, $'\mathrm{G}$, $'\mathrm{H}$, instead of α, β, γ, F, G, H, and, for the present, supposing $i+1$ to be positive*, we have the complete harmonic expressions of $'\mathrm{F}$, $'\mathrm{G}$, $'\mathrm{H}$, each in two terms, of orders i and $i+2$ respectively. Hence the A, A′, &c. of (45) are given by the following equations:—

$$\left.\begin{aligned}
\frac{\mathrm{A}_i}{a^{i+1}}&=\frac{\mathrm{A}'_i}{a'^{i+1}}=-\frac{m+(i+1)n}{(2i+3)(m+n)}r^{-i}\frac{d\mathrm{W}_{i+1}}{dx},\\
\frac{\mathrm{B}_i}{a^{i+1}}&=\frac{\mathrm{B}'_i}{a'^{i+1}}=-\frac{m+(i+1)n}{(2i+3)(m+n)}r^{-i}\frac{d\mathrm{W}_{i+1}}{dy},\\
\frac{\mathrm{C}_i}{a^{i+1}}&=\frac{\mathrm{C}'_i}{a'^{i+1}}=-\frac{m+(i+1)n}{(2i+3)(m+n)}r^{-i}\frac{d\mathrm{W}_{i+1}}{dz},\\
\frac{\mathrm{A}_{i+2}}{a^{i+1}}&=\frac{\mathrm{A}'_{i+2}}{a'^{i+1}}=\frac{(2i+5)m-n}{(2i+3)(2i+5)(m+n)}r^{i+3}\frac{d(\mathrm{W}_{i+1}r^{-2i-3})}{dx},\\
\frac{\mathrm{B}_{i+2}}{a^{i+1}}&=\frac{\mathrm{B}'_{i+2}}{a'^{i+1}}=\frac{(2i+5)m-n}{(2i+3)(2i+5)(m+n)}r^{i+3}\frac{d(\mathrm{W}_{i+1}r^{-2i-3})}{dy},\\
\frac{\mathrm{C}_{i+2}}{a^{i+1}}&=\frac{\mathrm{C}'_{i+2}}{a'^{i+1}}=\frac{(2i+5)m-n}{(2i+3)(2i+5)(m+n)}r^{i+3}\frac{d(\mathrm{W}_{i+1}r^{-2i-3})}{dz}.
\end{aligned}\right\}\quad . \quad . \quad . \quad (76)$$

46. The functions derived from A_i, B_i, C_i, &c., which are required for formulæ (48) and (49), are therefore as follows:—

$$\left.\begin{aligned}
&\frac{d(\mathrm{A}_ir^i)}{dx}+\frac{d(\mathrm{B}_ir^i)}{dy}+\frac{d(\mathrm{C}_ir^i)}{dz}=0,\\
&\frac{d(\mathrm{A}_ir^{-i-1})}{dx}+\frac{d(\mathrm{B}_ir^{-i-1})}{dy}+\frac{d(\mathrm{C}_ir^{-i-1})}{dz}=\frac{(i+1)(2i+1)[m+(i+1)n]}{(2i+3)(m+n)}\frac{a^{i+1}}{r^{2i+3}}\mathrm{W}_{i+1},\\
&\frac{d(\mathrm{A}_{i+2}r^{i+2})}{dx}+\frac{d(\mathrm{B}_{i+2}r^{i+2})}{dy}+\frac{d(\mathrm{C}_{i+2}r^{i+2})}{dy}=-\frac{(i+2)[(2i+5)m-n]}{(2i+3)(m+n)}a^{i+1}\mathrm{W}_{i+1},\\
&\frac{d(\mathrm{A}_{i+2}r^{-i-3})}{dx}+\frac{d(\mathrm{B}_{i+2}r^{-i-3})}{dy}+\frac{d(\mathrm{C}_{i+2}r^{-i-3})}{dz}=0,
\end{aligned}\right\}. \quad (77)$$

with the corresponding expressions relative to A'_i, B'_i, C'_i, &c., obtained simply by changing a into a'.

Hence by (48) and (50), and the two corresponding equations for the inner surface, we infer that each of the four functions ψ_{i-1}, ψ_{-i}, ϕ_{i-1}, ϕ_{-i} vanishes. By the same equations, with i changed into $i+2$, we obtain expressions, all of one harmonic form, direct or reciprocal, as follows, for the four functions of order $i+1$:—

* As we shall not in the present paper consider particularly any case of a shell influenced by centres of force in the hollow space within it, which alone could give a potential W_{i+1} of negative degree, we need not write any of the expressions in forms convenient for making $i+1$ negative.

$$\left.\begin{aligned}\psi_{i+1}&=K_{i+1}W_{i+1},\\ \psi_{-i-2}&=K'_{i+1}r^{-2i-3}W_{i+1},\\ \phi_{i+1}&=L_{i+1}W_{i+1},\\ \phi_{-i-2}&=L'_{i+1}r^{-2i-3}W_{i+1},\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(78)$$

K_{i+1}, K'_{i+1}, L_{i+1}, L'_{i+1}, which need not be here explicitly expressed, being four constants obtained from the solution of four simple algebraic equations. Lastly, by the four equations with $(i+4)$ instead of i, we find that ψ_{i+3}, ψ_{-i-4}, ϕ_{i+3}, ϕ_{-i-4} all vanish. Using these results for ψ and ϕ in (47), we see that each of the functions u must be a harmonic congruent with either $\frac{dW_{i+1}}{dx}$ or $\frac{d(W_{i+1}r^{-2i-3})}{dx}$. Hence, by using (78) in (38) and (39) we find

$$\left.\begin{aligned}u_i&=\frac{-L_{i+1}}{(2i+1)(i+1)}\frac{dW_{i+1}}{dx}, & u_{-i-1}&=\frac{-K'_{i+1}}{(2i+1)(i+1)}r^{-2i-1}\frac{dW_{i+1}}{dx},\\ u_{i+2}&=\frac{-K_{i+1}}{(2i+5)(i+2)}r^{2i+5}\frac{d(W_{i+1}r^{-2i-3})}{dx}, & u_{-i-3}&=\frac{-L'_{i+1}}{(2i+5)(i+2)}\frac{d(W_{i+1}r^{-2i-3})}{dx};\end{aligned}\right\}\quad(79)$$

and symmetrical expressions for v and w. Finally, using these expressions, (79) and (78), in (36), and the result in (69) with (73), we arrive at an explicit solution of the problem in the following remarkably simple form:—

$$\left.\begin{aligned}\alpha&=\mathfrak{E}_{i+1}\frac{dW_{i+1}}{dx}+\mathfrak{E}'_{i+1}\frac{d(W_{i+1}r^{-2i-3})}{dx},\\ \beta&=\mathfrak{E}_{i+1}\frac{dW_{i+1}}{dy}+\mathfrak{E}'_{i+1}\frac{d(W_{i+1}r^{-2i-3})}{dy},\\ \gamma&=\mathfrak{E}_{i+1}\frac{dW_{i+1}}{dz}+\mathfrak{E}'_{i+1}\frac{d(W_{i+1}r^{-2i-3})}{dz},\end{aligned}\right\}\quad\ldots\ldots\ldots(80)$$

where

$$\left.\begin{aligned}\mathfrak{E}_{i+1}&=-\frac{L_{i+1}+K'_{i+1}r^{-2i-1}}{(i+1)(2i+1)}+\left\{\frac{1}{2(2i+3)(m+n)}-\tfrac{1}{2}m\frac{K_{i+1}}{(m+2n)i+m+3n}\right\}r^2,\\ \mathfrak{E}'_{i+1}&=-\frac{K_{i+1}r^{2i+5}+L'_{i+1}}{(i+2)(2i+5)}+\left\{\frac{-r^{2i+5}}{(2i+3)(2i+5)(m+n)}+\frac{mr^2}{2}\frac{K'_{i+1}}{(m+2n)i+2m+3n}\right\}.\end{aligned}\right\}\quad.\ (81)$$

47. In conclusion, let us consider the case of a solid sphere. For this we have

$$\psi_{-i-2}=0,\ \text{and}\ \phi_{-i-2}=0,$$

as we see at once from the character of the problem, or as we find by putting $a'=0$ in the four equations by which in § 46 we have seen that K_{i+1}, K'_{i+1}, L_{i+1}, L'_{i+1} are to be determined. Then, by (48), with i changed into $i+2$, and by (49), we find

$$\left.\begin{aligned}\psi_{i+1}&=-\frac{(i+2)[(m+2n)i+m+3n][m(2i+5)-n]}{(2i+3)n\{m[2(i+2)^2+1]-n(2i+3)\}(m+n)}W_{i+1},\\ \phi_{i+1}&=a^2\frac{(i+1)^2(2i+1)[m(i+3)-n]}{2n\{m[2(i+2)^2+1]-n(2i+3)\}}W_{i+1}.\end{aligned}\right\}\quad\ldots\ldots(82)$$

The coefficients of W_{i+1} in these expressions are the values which we must take for K_{i+1} and L_{i+1} respectively in (81); and therefore, after reductions which show $(m+n)$ as a factor of the numerator of each fraction in which it appears at first as a factor of the denominator, we have

$$\left.\begin{aligned}\mathfrak{E}_{i+1}&=-\frac{(i+1)[m(i+3)-n]a^2}{2n\{m[2(i+2)^2+1]-n(2i+3)\}}+\frac{[(i+2)(2i+5)m-(2i+3)n]r^2}{2n(2i+3)\{m[2(i+2)^2+1]-n(2i+3)\}},\\ \mathfrak{E}'_{i+1}&=\frac{(i+1)mr^{2i+5}}{n(2i+3)\{m[2(i+2)^2+1]-n(2i+3)\}}.\end{aligned}\right\} \quad (83)$$

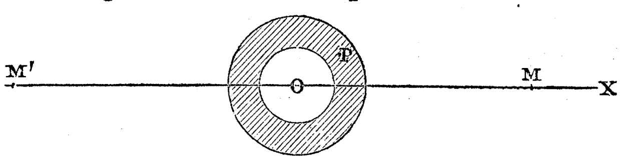
These, substituted in (80), give expressions for α, β, γ which constitute a complete and explicit solution of the problem.

It is easy to verify this result, by testing that (56) (with $-X=\frac{dW_{i+1}}{dx}$, &c.) is satisfied for every point of the solid, and that equations (34) give $F=0$, $G=0$, $H=0$ at the bounding surface, $r=a$.

48. The case of $i=1$ is, as we shall immediately see, of high importance. For it the preceding expressions, (83) and (80), become

$$\left.\begin{aligned}\mathfrak{E}_2&=\frac{-10(4m-n)a^2+(21m-5n)r^2}{10n(19m-5n)},\\ \mathfrak{E}'_2&=\frac{4mr^7}{10n(19m-5n)},\\ \alpha&=\mathfrak{E}_2\frac{dW_2}{dx}+\mathfrak{E}'_2\frac{d(W_2r^{-5})}{dx},\\ \beta&=\mathfrak{E}_2\frac{dW_2}{dy}+\mathfrak{E}'_2\frac{d(W_2r^{-5})}{dy},\\ \gamma&=\mathfrak{E}_2\frac{dW_2}{dz}+\mathfrak{E}'_2\frac{(dW_2r^{-5})}{dz}.\end{aligned}\right\} \quad \dots\dots\dots (84)$$

49. As an example of the application of §§ 45–48, let us suppose a spherical shell or solid sphere to be equilibrated under the influence of masses collected in two fixed external points*, and each attracting according to the inverse square of its distance. Let the two masses M, M′ be in the axis OX; and, P being the point whose coordinates are x, y, z, let PM=D, PM′=D′. Let also OM=c, OM′=c'.



Then, if m, m′ denote the two masses, for equilibrium we must have

$$\frac{m}{c^2}=\frac{m'}{c'^2}.$$

* If our limits permitted, a highly interesting example might be made of the case of a shell under the influence of a single attracting point in the hollow space within it. The effect will clearly be to keep the whole shell sensibly in equilibrium even if the attracting point is excentric; and under stress even if the attracting point is in the centre.

The potential at P, due to the two masses, will be $\frac{\mathrm{m}}{\mathrm{D}}+\frac{\mathrm{m}'}{\mathrm{D}'}$, or, according to the notation of § 42, with, besides, w taken to denote the mass of unit volume of the elastic solid,

$$-\mathrm{W}=w\left(\frac{\mathrm{m}}{\mathrm{D}}+\frac{\mathrm{m}'}{\mathrm{D}'}\right).$$

The known forms in the elementary theory of spherical harmonics give immediately the development of this in a converging infinite series of solid harmonic terms. We have only then to apply the solution of §§ 45, 46 to each term, to obtain a series expressing the required solution.

50. We may work out this result explicitly for the case in which both masses are very distant; and for simplicity we shall suppose one of them infinitely more distant than the other; that is to say, we shall suppose it to exercise merely a constant balancing force on the substance of the shell. We shall then have precisely the same bodily disturbing force as that which the earth experiences from the moon alone, or from the sun alone.

51. Referring to the diagram and notation of § 49, we have

$$\frac{1}{\mathrm{D}}=\frac{1}{c}\left\{1+\frac{x}{c}+\frac{x^2-\frac{1}{2}(y^2+z^2)}{c^2}\right\}$$

if we neglect higher powers of $\frac{x}{c}, \frac{y}{c}, \frac{z}{c}$ than the square; and

$$\frac{1}{\mathrm{D}'}=\frac{1}{c'}\left(1-\frac{x}{c'}\right)$$

neglecting all higher powers of $\frac{x}{c}, \frac{y}{c}, \frac{z}{c}$. Hence, taking account of the relation $\frac{\mathrm{m}}{c^2}=\frac{\mathrm{m}'}{c'^2}$ required for equilibrium, we have, for the disturbance potential,

$$-\mathrm{W}=\frac{\mathrm{m}}{c^3}(x^2-\tfrac{1}{2}y^2-\tfrac{1}{2}z^2)w,$$

an irrelevant constant being omitted from the expression which § 49 would give. This being a harmonic of the second degree, we may use it for W_{i+1}, putting $i=1$ in the formulæ of § 47, and thus solve the problem of finding the deformation of a homogeneous spherical shell under the influence of a distant attracting mass and a uniform balancing force. I hope, in a future communication to the Royal Society, to show the application of this result to the case of the lunar and solar influence on a body such as the earth is assumed to be by many geologists—that is to say, a solid crust, constituting a spheroidal shell, of some thickness less than 100 miles, with its interior filled with liquid. The untenability of this hypothesis is, however, sufficiently demonstrated by the considerations adduced in a previous communication ("On the Rigidity of the Earth," read May 8, 1862), in which the following explicit solution of the problem for a homogeneous solid sphere only is used.

52. Using the expression of § 51 for W_2, we have

$$\left.\begin{aligned}
&\frac{dW_2}{dx}=-2\frac{\mathrm{m}}{c^3}xw,\\
&\frac{dW_2}{dy}=+\frac{\mathrm{m}}{c^3}yw,\quad \frac{dW_2}{dz}=+\frac{\mathrm{m}}{c^3}zw,\\
&\frac{d(W_2r^{-5})}{dx}=+3\frac{\mathrm{m}}{c^3}\frac{(x^2-\frac{3}{2}y^2-\frac{3}{2}z^2)x}{r^{-7}}w,\\
&\frac{d(W_2r^{-5})}{dy}=+3\frac{\mathrm{m}}{c^3}\frac{(2x^2-\frac{1}{2}y^2-\frac{1}{2}z^2)y}{r^{-7}}w,\quad \frac{d(W_2r^{-5})}{dz}=+3\frac{\mathrm{m}}{c^3}\frac{(2x^2-\frac{1}{2}y^2-\frac{1}{2}z^2)z}{r^7}w.
\end{aligned}\right\}\quad . \quad (85)$$

These formulæ being substituted for the differential coefficients which appear in (84), we have algebraic expressions for the displacement of any point of the solid.

The condition of the body being symmetrical about the axis of x, we may conveniently assume

$$y=\mathrm{y}\cos\phi,\quad z=\mathrm{y}\sin\phi,$$

$$\beta^2+\gamma^2=\mu^2;$$

so that we shall have (as we see by the preceding expressions)

$$\beta=\mu\cos\phi,$$

$$\gamma=\mu\sin\phi,$$

and μ will denote the component displacement perpendicular to OX. If, further, we assume

$$x=r\cos\theta,$$

$$\mathrm{y}=r\sin\theta,$$

the expressions (84) for the component displacements, with (85) used in them, give

$$\left.\begin{aligned}
\alpha&=w\frac{\mathrm{m}}{c^3}\left\{-2r\mathfrak{E}_2+\frac{3}{2}\frac{\mathfrak{E}_2'}{r^4}(5\cos^2\theta-3)\right\}\cos\theta,\\
\mu&=w\frac{\mathrm{m}}{c^3}\left\{\quad r\mathfrak{E}_2+\frac{3}{2}\frac{\mathfrak{E}_2'}{r^4}(5\cos^2\theta-1)\right\}\sin\theta.
\end{aligned}\right\}\quad . \; . \; . \; . \; . \; . \; . \quad (86)$$

The values given in (84) for $\mathfrak{E}_2$ and $\mathfrak{E}_2'$ are to be used for any internal point, at a distance r from the centre, in these equations (86), and thus we have the simplest possible expression for the required displacement of any point of the solid.

53. If we resolve the displacement along and perpendicular to the radius, and consider only the radial component, we see that the series of concentric spherical surfaces of the undisturbed globe become spheroids of revolution in the distorted body. The elongation of the axial radius, obtained by putting $\theta=0$ and taking the value of α, is double the shortening of the equatorial radius, obtained by putting $\theta=\frac{1}{2}\pi$ and taking the value of μ; which we might have inferred from the fact, shown by the general equations (80) above, that there can be no alteration of volume on the whole within any one of these surfaces. The expression for the excess of the axial above the equatorial radius is

$$-r\mathfrak{E}_2+\frac{9}{2}\frac{\mathfrak{E}_2'}{r^4},$$

which, if we substitute for $\mathfrak{E}_2$ and $\mathfrak{E}'_2$ their values by (84), becomes

$$3w\frac{\mathrm{m}}{c^3}\frac{2(4m-n)a^2-(3m-n)r^2}{10n(19m-5n)}r.$$

If in this we take $r=a$, and $m=\infty$, it becomes $\frac{2}{3}\frac{\mathrm{m}}{c^3}\frac{5}{19n}r^3$, which is the result used in § 34 of the paper on the Rigidity of the Earth, preceding the present in the Transactions.

54. In the case of $a'=0$, the result of § 18 takes the extremely simple form

$$\left.\begin{aligned}
\alpha&=\Sigma\left\{\mathrm{A}_i\left(\frac{r}{a}\right)^i+\frac{m(a^2-r^2)}{2a^i[n(2i-1)+m(i-1)]}\frac{d\Theta_{i-1}}{dx}\right\},\\
\beta&=\Sigma\left\{\mathrm{B}_i\left(\frac{r}{a}\right)^i+\frac{m(a^2-r^2)}{2a^i[n(2i-1)+m(i-1)]}\frac{d\Theta_{i-1}}{dy}\right\},\\
\gamma&=\Sigma\left\{\mathrm{C}_i\left(\frac{r}{a}\right)^i+\frac{m(a^2-r^2)}{2a^i[n(2i-1)+m(i-1)]}\frac{d\Theta_{i-1}}{dz}\right\},
\end{aligned}\right\} \quad \ldots\ldots \quad (87)$$

where

$$\Theta_{i-1}=\frac{d(\mathrm{A}_ir^i)}{dx}+\frac{d(\mathrm{B}_ir^i)}{dy}+\frac{d(\mathrm{C}_ir^i)}{dz}.$$

This expresses the displacement at any point within a solid sphere of radius a, when its surface is displaced in a given manner ($\Sigma\mathrm{A}_i$, $\Sigma\mathrm{B}_i$, $\Sigma\mathrm{C}_i$). And merely by making i negative we have, in the same formula, the solution of the same problem for an infinite solid with a hollow spherical space every point of the surface of which is displaced to a given distance in a given direction. These solutions are obtained directly, with great ease, by the method of §§ 6–15, or are easily proved by direct verification, without any of the intricacy of analysis inevitable when, as in the general investigations with which we commenced, a shell bounded by two concentric spherical surfaces is the subject.

[Added since the reading of the Paper.]

§§ 55 to 58. *Oscillations of a Liquid Sphere.*

55. Let V be the gravitation potential at any point P(x, y, z), and h the height of the surface (or radial component of its displacement) from the mean spherical surface at a point E in the radius through P. Then, if

$$h=\mathrm{S}_1+\mathrm{S}_2+\ldots \quad \ldots\ldots \quad (88)$$

be the expression for h in terms of spherical surface harmonic functions of the position of E, and if μ be the attraction on the unit of mass exercised by a particle equal in mass to the unit bulk of the liquid, we have, by the known methods for finding the attractions of bodies infinitely nearly spherical (THOMSON and TAIT's 'Natural Philosophy,' chap. vi.),

$$\left.\begin{aligned}
\mathrm{V}&=4\pi a\mu\left\{\tfrac{1}{2}a-\tfrac{1}{6}\frac{r^2}{a}+\Sigma\left(\frac{r}{a}\right)^i\frac{\mathrm{S}_i}{2i+1}\right\} && \text{when } r<a,\\
\mathrm{V}&=4\pi a\mu\left\{\tfrac{1}{3}\frac{a^2}{r}+\frac{a}{r}\Sigma\left(\frac{a}{r}\right)^i\frac{\mathrm{S}_i}{2i+1}\right\} && \quad ,, \quad r>a,\\
\mathrm{V}&=4\pi a\mu\left\{\tfrac{1}{3}a+\Sigma\frac{\mathrm{S}_i}{2i+1}\right\} && \quad ,, \quad r=a.
\end{aligned}\right\} \quad \ldots\ldots \quad (89)$$

and

In these

$$4\pi\mu a = 3g, \quad \ldots \ldots \quad (90)$$

if g denote the force of gravity at the surface, due to the mean sphere, of radius a.

56. Now for infinitely small motions the ordinary kinetic equations give

$$-\frac{dp}{dx}=\varrho\left(\frac{du}{dt}-\frac{dV}{dx}\right); \quad -\frac{dp}{dy}=\varrho\left(\frac{dv}{dt}-\frac{dV}{dy}\right); \quad -\frac{dp}{dz}=\varrho\left(\frac{dw}{dt}-\frac{dV}{dz}\right); \quad \ldots \quad (91)$$

where ϱ is the mass per unit of volume; u, v, w the component velocities through the fixed point P at time t; and p the fluid pressure. Hence, possible non-periodic motions being omitted, $udx+vdy+wdz$ is a complete differential; and, denoting it by ϕ, d we have

$$C-p=\varrho\left(\frac{d\phi}{dt}-V\right). \quad \ldots \ldots \quad (92)$$

57. To find the surface conditions,—first, since the pressure has a constant value, Π, at the free surface,

$$p=g\varrho h+\Pi \text{ when } r=a, \quad \ldots \ldots \quad (93)$$

the variations of gravity depending on the variations of figure being of course neglected in the infinitely small term $g\varrho h$. And, since $\frac{dh}{dt}$ is the radial component of the velocity at E, we have, when $r=a$,

$$\frac{x}{r}\frac{d\phi}{dx}+\frac{y}{r}\frac{d\phi}{dy}+\frac{z}{r}\frac{d\phi}{dz}=\frac{dh}{dt}. \quad \ldots \ldots \quad (94)$$

Now since, the fluid being incompressible, $\nabla^2\phi=0$, ϕ may be expanded in a series of solid harmonic functions; let

$$\phi=\Sigma\Phi_i\left(\frac{r}{a}\right)^i, \quad \ldots \ldots \quad (95)$$

where $\Phi_1, \Phi_2, \ldots$ are surface harmonics. Hence, as the successive terms are homogeneous functions of the coordinates (x, y, z), of degrees 1, 2, &c.,

$$\frac{x}{r}\frac{d\phi}{dx}+\frac{y}{r}\frac{d\phi}{dy}+\frac{z}{r}\frac{d\phi}{dz}=\frac{1}{r}\Sigma i\Phi_i\left(\frac{r}{a}\right)^i, \quad \ldots \ldots \quad (96)$$

and therefore, by (88) and (94),

$$\frac{dS_i}{dt}=\frac{i}{a}\Phi_i. \quad \ldots \ldots \quad (97)$$

58. Eliminating p between (92) with $r=a$ and (93), substituting for V by (89) and (90), differentiating, substituting for $\frac{dS_i}{dt}$ by (97), and comparing harmonic terms of order i, we have

$$\frac{d^2\Phi_i}{dt^2}+\frac{g}{a}i\left(1-\frac{3}{2i+1}\right)\Phi_i; \quad \ldots \ldots \quad (98)$$

of which the integral is

$$\Phi_i=A\cos\left\{t\sqrt{\frac{g}{a}i\left(1-\frac{3}{2i+1}\right)}-E\right\}. \quad \ldots \ldots \quad (99)$$

Here A is a surface spherical harmonic function of the coordinates of E expressing the maximum value of Φ_i, and E is the epoch (THOMSON and TAIT, § 53) of the simple harmonic function of the time which we find to represent Φ_i. Using this solution in (97) and (88), we see that if the surface be normally displaced according to a spherical harmonic of order i, and left to itself, the resulting motion gives rise to a simple harmonic variation of the normal displacement, having for period

$$2\pi\sqrt{\frac{a}{g}\,\frac{2i+1}{2i(i-1)}},$$

that is, the period of a common pendulum of length $\frac{(2i+1)a}{2i(i-1)}$. It is worthy of remark that the period of vibration thus calculated is the same for the same density of liquid, whatever be the dimensions of the globe.

For the case of $i=2$, or an ellipsoidal deformation, the length of the isochronous pendulum becomes $\frac{5}{4}a$, or one and a quarter times the earth's radius, for a homogeneous liquid globe of the same mass and diameter as the earth; and therefore for this case, or for any homogeneous liquid globe of about $5\frac{1}{2}$ times the density of water, the half-period is $47^{\mathrm{m}}\,12^{\mathrm{s}}$, which is the result stated in the paper "On the Rigidity of the Earth" (§ 3), preceding the present in the Transactions.

APPENDIX, §§ 59–71.—*General Theory of the Equilibrium of an Elastic Solid.*

59. Let a solid composed of matter fulfilling no condition of isotropy in any part, and not homogeneous from part to part, be given of any shape, unstrained, and let every point of its surface be altered in position to a given distance in a given direction. It is required to find the displacement of every point of its substance in equilibrium. Let x, y, z be the coordinates of any particle, P, of the substance in its undisturbed position, and $x+\alpha, y+\beta, z+\gamma$ its coordinates when displaced in the manner specified; that is to say, let α, β, γ be the components of the required displacement. Then, if for brevity we put

$$\left.\begin{aligned}
A&=\left(\frac{d\alpha}{dx}+1\right)^2+\left(\frac{d\beta}{dx}\right)^2+\left(\frac{d\gamma}{dx}\right)^2,\\
B&=\left(\frac{d\alpha}{dy}\right)^2+\left(\frac{d\beta}{dy}+1\right)^2+\left(\frac{d\gamma}{dy}\right)^2,\\
C&=\left(\frac{d\alpha}{dz}\right)^2+\left(\frac{d\beta}{dz}\right)^2+\left(\frac{d\gamma}{dz}+1\right)^2,\\
a&=\frac{d\alpha}{dy}\frac{d\alpha}{dz}+\left(\frac{d\beta}{dy}+1\right)\frac{d\beta}{dz}+\frac{d\gamma}{dy}\left(\frac{d\gamma}{dz}+1\right),\\
b&=\frac{d\alpha}{d}\left(\frac{d\alpha}{dx}+1\right)+\frac{d\beta}{dz}\frac{d\beta}{dx}+\left(\frac{d\gamma}{dz}+1\right)\frac{d\gamma}{dx},\\
c&=\left(\frac{d\alpha}{dx}+1\right)\frac{d\alpha}{dy}+\frac{d\beta}{dx}\left(\frac{d\beta}{dy}+1\right)+\frac{d\gamma}{dx}\frac{d\gamma}{dy};
\end{aligned}\right\} \quad \dots\dots \quad (100)$$

these six quantities A, B, C, a, b, c, as is known*, thoroughly determine the strain experienced by the substance infinitely near the particle P (irrespectively of any rotation it may experience), in the following manner:—

* THOMSON and TAIT'S 'Natural Philosophy,' § 190 (e) and § 181 (5).

60. Let ξ, η, ζ be the undisturbed coordinates of a particle infinitely near P, relatively to axes through P parallel to those of x, y, z respectively; and let $\xi_{,}, \eta_{,}, \zeta_{,}$ be the coordinates, relative still to axes through P, when the solid is in its strained condition. Then

$$\xi_{,}^2+\eta_{,}^2+\zeta_{,}^2=\text{A}\xi^2+\text{B}\eta^2+\text{C}\zeta^2+2a\eta\zeta+2b\zeta\xi+2c\xi\eta\,; \quad \ldots \quad (101)$$

and therefore all particles which in the strained state lie on a spherical surface

$$\xi_{,}^2+\eta_{,}^2+\zeta_{,}^2=r_{,}^2,$$

are, in the unstrained state, on the ellipsoidal surface,

$$\text{A}\xi^2+\text{B}\eta^2+\text{C}\zeta^2+2a\eta\zeta+2b\zeta\xi+2c\xi\eta=r_{,}^2.$$

This, as is well known*, completely defines the homogeneous strain of the matter in the neighbourhood of P.

61. Hence the thermo-dynamic principles by which, in a paper on the Thermo-elastic Properties of Matter in the first Number of the 'Quarterly Mathematical Journal' (April 1855), GREEN's dynamical theory of elastic solids was demonstrated as part of the modern dynamical theory of heat, show that if $w.dxdydz$ denote the work required to alter an infinitely small undisturbed volume, $dxdydz$, of the solid, into its disturbed condition, when its temperature is kept constant, we must have

$$w=f(\text{A, B, C}, a, b, c), \quad \ldots \quad (102)$$

where f denotes a positive function of the six elements, which vanishes when $\text{A}-1$, $\text{B}-1$, $\text{C}-1$, a, b, c each vanish. And if W denote the whole work required to produce the change actually experienced by the whole solid, we have

$$\text{W}=\iiint w dxdydz, \quad \ldots \quad (103)$$

where the triple integral is extended through the space occupied by the undisturbed solid.

62. The position assumed by every particle in the interior of the solid will be such as to make this a minimum, subject to the condition that every particle of the surface takes the position given to it, this being the elementary condition of stable equilibrium. Hence, by the method of variation,

$$\delta\text{W}=\iiint \delta w dxdydz=0. \quad \ldots \quad (104)$$

But, exhibiting only terms depending on $\delta\alpha$, we have

$$\delta w=\left\{2\frac{dw}{d\text{A}}\left(\frac{d\alpha}{dx}+1\right)+\frac{dw}{db}\frac{d\alpha}{dz}+\frac{dw}{dc}\frac{d\alpha}{dy}\right\}\frac{d\delta\alpha}{dx}$$
$$+\left\{2\frac{dw}{d\text{B}}\frac{d\alpha}{dy}+\frac{dw}{da}\frac{d\alpha}{dz}+\frac{dw}{dc}\left(\frac{d\alpha}{dx}+1\right)\right\}\frac{d\delta\alpha}{dy}$$
$$+\left\{2\frac{dw}{d\text{C}}\frac{d\alpha}{dz}+\frac{dw}{da}\frac{d\alpha}{dy}+\frac{dw}{db}\left(\frac{d\alpha}{dx}+1\right)\right\}\frac{d\delta\alpha}{dz}$$
$$+\&\text{c}.$$

* THOMSON and TAIT's 'Natural Philosophy,' §§ 155–165.

Hence, integrating by parts, and observing that $\delta\alpha$, $\delta\beta$, $\delta\gamma$ vanish at the limiting surface, we have

$$\delta W = -\iiint dxdydz\left\{\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\delta\alpha+\&c.\right\}, \quad . \quad . \quad . \quad . \quad (105)$$

where for brevity P, Q, R denote the factors of $\frac{d\delta\alpha}{dx}$, $\frac{d\delta\alpha}{dy}$, $\frac{d\delta\alpha}{dz}$ respectively, in the preceding expression. In order that δW may vanish, the factors of $\delta\alpha$, $\delta\beta$, $\delta\gamma$ in the expression now found for it must each vanish; and hence we have, as the equations of equilibrium,

$$\left.\begin{array}{l} \frac{d}{dx}\left\{2\frac{dw}{dA}\left(\frac{d\alpha}{dx}+1\right)+\frac{dw}{db}\frac{d\alpha}{dz}+\frac{dw}{dc}\frac{d\alpha}{dy}\right\} \\ \quad +\frac{d}{dy}\left\{2\frac{dw}{dB}\frac{d\alpha}{dy}+\frac{dw}{da}\frac{d\alpha}{dz}+\frac{dw}{dc}\left(\frac{d\alpha}{dx}+1\right)\right\} \\ \quad +\frac{d}{dz}\left\{2\frac{dw}{dC}\frac{d\alpha}{dz}+\frac{dw}{da}\frac{d\alpha}{dy}+\frac{dw}{db}\left(\frac{d\alpha}{dx}+1\right)\right\}=0, \\ \qquad \&c. \ \&c., \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

of which the second and third, not exhibited, may be written down merely by attending to the symmetry.

63. From the property of w that it is necessarily positive when there is any strain, it follows that there must be some distribution of strain through the interior which shall make $\iiint w dxdydz$ *the least possible*, subject to the prescribed surface condition, and therefore that the solution of equations (106), subject to this condition, is possible. If, whatever be the nature of the solid as to difference of elasticity in different directions, in any part, and as to heterogeneousness from part to part, and whatever be the extent of the change of form and dimensions to which it is subjected, there cannot be any internal configuration of unstable equilibrium, or consequently any but one of stable equilibrium, with the prescribed surface displacement and no disturbing force on the interior, then, besides being always positive, w must be such a function of A, B, &c. that there can be only one solution of the equations. This is obviously the case when the unstrained solid is homogeneous.

64. It is easy to include, in a general investigation similar to the preceding, the effects of any force on the interior substance, such as we have considered particularly for a spherical shell, of homogeneous isotropic matter, in §§ 35–46 above. It is also easy to adapt the general investigation to superficial data of *force*, instead of displacement.

65. Whatever be the general form of the function f for any part of the substance, since it is always positive it cannot change in sign when A−1, B−1, C−1, a, b, c have their signs changed; and therefore for infinitely small values of these quantities it must be a homogeneous quadratic function of them with constant coefficients. (And it may be useful to observe that for all values of the variables A, B, &c., it must therefore be expressible in the same form, with varying coefficients, each of which is always finite, for

all values of the variables.) Thus, for infinitely small strains, we have GREEN'S theory of elastic solids, founded on a homogeneous quadratic function of the components of strain, expressing the work required to produce it. Putting

$$A-1=2e,\ B-1=2f,\ C-1=2g, \quad \dots \dots \quad (107)$$

and denoting by $\frac{1}{2}(e, e)$, $\frac{1}{2}(f, f)$, ... (e, f), ... (e, a), ... the coefficients, we have

$$\left.\begin{aligned} w=\tfrac{1}{2}\{(e, e)e^2+(f, f)f^2+(g, g)g^2+(a, a)a^2+(b, b)b^2+(c, c)c^2\} & \\ +(e, f)ef+(e, g)eg+(e, a)ea+(e, b)eb+(e, c)ec & \\ +(f, g)fg+(f, a)fa+(f, b)fb+(f, c)fc & \\ +(g, a)ga+(g, b)gb+(g, c)gc & \\ +(a, b)ab+(a, c)ac & \\ +(b, c)bc & \end{aligned}\right\} \quad . \ . \quad (108)$$

The twenty-one coefficients in this expression constitute the twenty-one coefficients of elasticity, which GREEN first showed to be proper and essential for a complete theory of the dynamics of an elastic solid subjected to infinitely small strains.

66. When the strains are infinitely small, the products $\frac{dw}{dA}\frac{d\alpha}{dx}$, $\frac{dw}{db}\frac{d\alpha}{dz}$, &c. are each infinitely small, of the second order. We therefore omit them; and then, attending to (107), we reduce (106) to

$$\left.\begin{aligned} \frac{d}{dx}\frac{dw}{de}+\frac{d}{dy}\frac{dw}{dc}+\frac{d}{dz}\frac{dw}{db}&=0,\\ \frac{d}{dx}\frac{dw}{dc}+\frac{d}{dy}\frac{dw}{df}+\frac{d}{dz}\frac{dw}{da}&=0,\\ \frac{d}{dx}\frac{dw}{db}+\frac{d}{dy}\frac{dw}{da}+\frac{d}{dz}\frac{dw}{dg}&=0, \end{aligned}\right\} \quad \dots \dots \quad (109)$$

which are the equations of interior equilibrium. Attending to (108) we see that $\frac{dw}{de}\dots\frac{dw}{da}\dots$ are linear functions of e, f, g, a, b, c the components of strain. Writing out one of them as an example, we have

$$\frac{dw}{de}=(e, e)e+(e, f)f+(e, g)g+(e, a)a+(e, b)b+(e, c)c. \quad . \ . \ . \quad (110)$$

And α, β, γ denoting, as before, the component displacements of any interior particle, P, from its undisturbed position $(x\, y, z)$, we have, by (107) and (100),

$$\left.\begin{aligned} e=\frac{d\alpha}{dx},\quad f=\frac{d\beta}{dy},\quad g=\frac{d\gamma}{dz},\\ a=\frac{d\beta}{dz}+\frac{d\gamma}{dy},\quad b=\frac{d\gamma}{dx}+\frac{d\alpha}{dz},\quad c=\frac{d\alpha}{dy}+\frac{d\beta}{dx}. \end{aligned}\right\} \quad \dots \dots \quad (111)$$

It is to be observed that the coefficients (e, e) (e, f), &c. will be in general functions of (x, y, z), but will be each constants when the unstrained solid is homogeneous.

67. It is now easy to prove directly, for the case of infinitely small strains, that the solution of the equations of interior equilibrium, whether for a heterogeneous or a homogeneous solid, subject to the prescribed surface condition, is unique. For let α, β, γ be components of displacement fulfilling the equations, and let α', β', γ' denote any other functions of (x, y, z) having the same surface values as α, β, γ, and let $e', f', \ldots, w'$ denote functions depending on them in the same way as $e, f, \ldots, w$ depend on α, β, γ. Thus, by TAYLOR'S theorem,

$$w'-w=\frac{dw}{de}(e'-e)+\frac{dw}{df}(f'-f)+\frac{dw}{dg}(g'-g)+\frac{dw}{da}(a'-a)+\frac{dw}{db}(b'-b)+\frac{dw}{dc}(c'-c)+\mathrm{H},$$

where H denotes the same homogeneous quadratic function of $e'-e$, &c. that w is of e, &c. If for $e'-e$, &c. we substitute their values by (111), this becomes

$$w'-w=\frac{dw}{de}\frac{d(\alpha'-\alpha)}{dx}+\frac{dw}{db}\frac{d(\alpha'-\alpha)}{dz}+\frac{dw}{dc}\frac{d(\alpha'-\alpha)}{dy}+\&\text{c.}+\mathrm{H}.$$

Multiplying by $dxdydz$, integrating by parts, observing that $\alpha'-\alpha$, $\beta'-\beta$, $\gamma'-\gamma$ vanish at the bounding surface, and taking account of (109), we find simply

$$\iiint(w'-w)dxdydz=\iiint \mathrm{H}dxdydz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

But H is essentially positive. Therefore every other interior condition than that specified by (α, β, γ), provided only it has the same bounding surface, requires a greater amount of work than w to produce it: and the excess is equal to the work that would be required to produce, from a state of no displacement, such a displacement as superimposed on (α, β, γ) would produce the other. And inasmuch as (α, β, γ) fulfil only the conditions of satisfying (110) and having the given surface values, it follows that no other than one solution can fulfil these conditions.

68. But (as has been remarked by Professor STOKES to the author) when the surface data are of force, not of displacement, or when force acts from without, on the interior substance of the body, the solution is not in general unique, and there may be configurations of unstable equilibrium, even with infinitely small displacement. For instance, let part of the body be composed of a steel bar magnet; and let a magnet be held outside in the same line, and with a pole of the same name in its end nearest to one end of the inner magnet. The equilibrium will be unstable, and there will be positions of stable equilibrium with the inner bar slightly inclined to the line of the outer bar, unless the rigidity of the rest of the body exceed a certain limit.

69. Recurring to the general problem, in which the strains are not supposed infinitely small, we see that, if the solid is isotropic in every part, the function of A, B, C, a, b, c which expresses w must be merely a function of the roots of the equation*

$$(\mathrm{A}-\zeta^2)(\mathrm{B}-\zeta^2)(\mathrm{C}-\zeta^2)-a^2(\mathrm{A}-\zeta^2)-b^2(\mathrm{B}-\zeta^2)-c^2(\mathrm{C}-\zeta^2)+2abc=0, \quad . \quad (113)$$

which (that is the positive values of ζ) are the ratios of elongation along the principal

* THOMSON and TAIT'S 'Natural Philosophy,' § 181 (11).

axes of the strain-ellipsoid. It is unnecessary here to enter on the analytical expression of this condition. For the case of A−1, B−1, C−1, a, b, c, each infinitely small, it obviously requires that

$$\left.\begin{array}{l}(e,e)=(f,f)=(g,g);\ (f,g)=(g,e)=(e,f);\ (a,a)=(b,b)=(c,c);\\ (e,a)=(f,b)=(g,c)=0;\ (b,c)=(c,a)=(a,b)=0;\\ \text{and}\quad (e,b)=(e,c)=(f,c)=(f,a)=(g,a)=(g,b)=0.\end{array}\right\} \quad . \quad (114)$$

Thus the twenty-one coefficients are reduced to three—

(e, e), which we may denote by the single letter $\mathfrak{A}$,
(f, g), " " " " $\mathfrak{B}$,
(a, a), " " " " n.

It is clear that this is necessary and sufficient for ensuring *cubic isotropy*—that is to say, perfect equality of elastic properties with reference to the three rectangular directions OX, OY, OZ. But for *spherical isotropy*, or complete isotropy with reference to all directions through the substance, it is further necessary that

$$\mathfrak{A}-\mathfrak{B}=2n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

as is easily proved analytically by turning two of the axes of coordinates in their own plane through 45°; or geometrically by examining the nature of the strain represented by any one of the elements a, b, c (a "simple shear") and comparing it with the resultant of c, and $f=-e$ (which is also a simple shear). It is convenient now to put

$$\mathfrak{A}+\mathfrak{B}=2m;\ \text{so that}\ \mathfrak{A}=m+n,\ \mathfrak{B}=m-n; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

and thus the expression for the potential energy per unit of volume becomes

$$2w=m(e+f+g)^2+n(e^2+f^2+g^2-2fg-2ge-2ef+a^2+b^2+c^2). \quad . \quad . \quad . \quad . \quad (117)$$

Using this in (108), and substituting for e, f, g, a, b, c their values by (111), we find immediately, for the equations of internal equilibrium, equations the same as (1) of § 2.

70. To find the mutual force exerted across any surface within the solid, as expressed by (26) of § 22, we have clearly, by considering the works done respectively by P, Q, R, S, T, U (§ 21) on any infinitely small change of figure or dimensions in the solid,

$$P=\frac{dw}{de},\ Q=\frac{dw}{df},\ R=\frac{dw}{dg},\ S=\frac{dw}{da},\ T=\frac{dw}{db},\ U=\frac{dw}{dc}. \quad . \quad . \quad . \quad . \quad (118)$$

Hence, for an isotropic solid, (117) gives the expression (25) of § 21, which we have used above.

71. To interpret the coefficients m and n in connexion with elementary ideas as to the elasticity of the solid; first let $a=b=c=0$, and $e=f=g=\frac{1}{3}\delta$; in other words, let the substance experience a uniform dilatation, in all directions, producing an expansion of volume from 1 to $1+\delta$. In this case (117) becomes

$$w=\tfrac{1}{2}(m-\tfrac{1}{3}n)\delta^2;$$

and we have

$$\frac{dw}{d\delta}=(m-\tfrac{1}{3}n)\delta.$$

Hence $(m-\frac{1}{3}n)\delta$ is the normal force per unit area of its surface required to keep any portion of the solid expanded to the amount specified by δ. Thus $m-\frac{1}{3}n$ measures the elastic force called out by, or the elastic resistance against, change of volume: and viewed as a *coefficient of elasticity*, it may be called the *elasticity of volume*. What is commonly called the "compressibility" is measured by $\frac{1}{m-\frac{1}{3}n}\cdot$

And let next $e=f=g=b=c=0$; which gives

$$w=\tfrac{1}{2}na^2;\text{ and, by (118), }S=na.$$

This shows that the tangential force per unit area required to produce an infinitely small shear*, amounting to a, is na. Hence n measures the innate power with which the body resists change of shape, and returns to its original shape when force has been applied to change it; that is to say, it measures *the rigidity* of the substance.

[NOTE added, December 1863].

Since this paper was communicated to the Royal Society, the author has found that the solution of the most difficult of the problems dealt with in it, which is the determination of the effect produced on a spherical shell by a prescribed application of force to its outer and inner surfaces, had previously been given by LAMÉ in a paper published in LIOUVILLE's Journal for 1854, under the title "Mémoire sur l'Équilibre l'élasticité des enveloppes sphériques." In the same paper LAMÉ shows how to take into account the effect of internal force, but does not solve the problem thus presented except for the simple cases of uniform gravity and of centrifugal force. The form in which the analysis has been applied in the present paper is very different from that chosen by LAMÉ (who uses throughout polar coordinates); but the principles are essentially the same, being merely those of spherical harmonic analysis, applied to problems presenting peculiar and novel difficulties.

* THOMSON and TAIT's 'Natural Philosophy,' § 171.